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THE UNIVERSITY OF ALBERTA

ON THE  $SU_3$  NUCLEAR SHELL MODEL

by

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A THESIS

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## ABSTRACT

Elliott's  $SU_3$  classification of nuclear shell model wave functions in LS coupling is extended and relationships between related formalisms are examined, with emphasis on the group-theoretical techniques.

To this end, the use of Young Diagrams to classify many-particle wave functions according to representations of various symmetric and linear groups is reviewed and illustrated. Operations such as inner and outer products and plethysms of representations of these different groups are redefined as operations on the Young Diagrams themselves, allowing the same data to be used with different interpretations for different groups. These techniques are used to review the supermultiplet classification of Wigner and Feenberg, as applied to harmonic oscillator wave functions in the nuclear shell model in LS coupling, with emphasis on the dual interpretation of this classification in terms of the unitary group  $SU_5$  as well as the symmetric group  $S_k$ . This classification and the corresponding supermultiplet model are applied to excited configurations.

Elliott's  $SU_3$  classification, (which extends to all shells in LS coupling an interesting property that the above-mentioned supermultiplet classification possesses in the p shell only, namely the property of predicting bands of eigenstates typical of the rotational model), is then introduced, exploiting an analogy with the  $SU_5$  classification: the  $r$ -quanta harmonic oscillator eigenfunctions, which correspond to homogeneous polynomials of degree  $r$  in three creation operators, are compared to the  $k$ -particle eigenfunctions, which are homogeneous polynomials of degree  $k$  in  $s$  variables, namely the  $s$  possible single-particle eigenfunctions. The  $SU_3$  model, using Moszkowski's quadrupole-quadrupole



interaction, is presented and it is shown that in the p shell its predictions are identical to those of the supermultiplet model provided the parameters  $\mathcal{L}$  and  $\mathcal{K}$  of the latter are suitably related to the 'moment of inertia' of the  $SU_3$  model. The  $SU_3$  model is then extended to excited configurations and applied to the experimental energy spectrum of  $Ne^{20}$ . A qualitative fit to the position of all fifteen observed states, belonging to five bands and to two or three configurations can be obtained with a single adjustable parameter. It is pointed out in particular that the  $SU_3$  model provides for a lowering of the excitation energy of the leading excited-configuration state below the value predicted by the unperturbed harmonic oscillator hamiltonian; this effect, which has no counterpart in the supermultiplet model, accounts for 40 % of the 10 Mev observed lowering in  $Ne^{20}$ .

Kretzschmar's alternative approach to the  $SU_3$  classification is then described and his technique for identifying spurious states is used to verify that the above-mentioned state of  $Ne^{20}$  is not spurious. Finally the pseudo-spin of Bargmann and Moshinsky is related to the reduction of a representation of  $SU_2$  introduced by Moshinsky as a generalization of his transformation brackets, and the formalism of Kretzschmar is used to explain the identity of the  $SU_A$  classification of Bargmann and Moshinsky to the  $SU_3$  classification of Elliott.





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## Chapter 1      GENERAL INTRODUCTION

### 1.1      Background and Motivation

This thesis is a contribution to the study of the  $SU_3$  classification of many-nucleon functions, which is a particular version of the LS-coupling shell model of nuclear structure. The first section of this introductory chapter outlines the background and motivation of the  $SU_3$  classification, as originally introduced in 1958. A second section lists the major contributions in this field since 1958 and a third explains the organization of the thesis.

There are two main reasons why nuclear structure theory is still an active research field. The atomic nucleus is an assembly of nucleons between which certain forces are acting, to be handled according to quantum mechanics. On one hand these basic forces themselves are not terribly well known or understood. On the other hand the indications are that if we knew more of the details we could not handle them anyway. Although certain recent developments are beginning to remove both limitations, the work discussed here stems from an earlier tradition. A frontal attack being impractical in the past, the natural approach has been to fall back on a study of models -- i.e. idealizations of the nucleus for which the calculations are manageable -- and to make the fullest possible use of general principles of symmetry and invariance. Thus models have played a leading role in nuclear theory, and group theory has been an important tool in their treatment.





For detailed reviews of these matters the reader is referred to articles by Moszkowski (Mz57) and by Elliott and Lane (E57), and for an up-to-date and more elementary treatment to Preston (Pr62).

Several different models have been developed to account for different portions of the large and growing body of experimental facts available, and a surprising situation has arisen: several of these models are too good to be abandoned. Thus it has become necessary to study not only the relations between models and experimental data, but also between model and model.

But there is not yet -- most people would agree -- a unique theory of nuclear structure. Such a theory would not necessarily make the models superfluous; it would not even, necessarily, make all but one model superfluous. But it should at least account in a coherent way for the respective ranges of applicability of various models to various phenomena. Several analytical tools have separately proven their worth; a synthesis is now needed, to fit all these partial explanations together. The  $SU_3$  work, it is generally agreed, has provided a significant step in the direction of such a synthesis.

It has done this by a technical improvement within the mathematical formalism of one of the important models, which has made it possible to obtain from that model certain results typical of another important model, hitherto considered as completely different from the first. The model within which improvements have been made is the independent particle model, or shell model; more precisely, the harmonic oscillator shell model with LS coupling. The model with which links have thus been established is the rotational model, part of the class of models referred to as collective.



Thus the  $SU_3$  work is not primarily an attempt to improve the detailed fit of theoretical predictions to experimental facts, and should not be judged on the basis of the small contributions it is making in that direction. To exaggerate a bit, it is not really a shell model of the nucleus, but a shell model of the rotational model of the nucleus. By way of contrast, the previous and highly successful 'unified model' due to Nilsson (N55) was also, in a very different sense, a synthesis of the harmonic oscillator shell model with the collective model. Essentially Nilsson combined the advantages of both by treating part of the nuclear problem in a collective way and part in an independent particle way: taking from the collective model the idea that some nuclei have a permanent deformation from spherical shape, Nilsson concluded that the average field felt by the individual nucleon is similarly deformed, and calculated single-particle states as a function of deformation. He was thus able to obtain an impressive improvement in detailed fits to experiment, which neither previous model had produced alone. Elliott's  $SU_3$  approach is not trying to compete with this: it attempts mostly to re-interpret known results rather than to improve agreement with experiment.

The general idea of the shell model is to emphasize the individual nucleons. In its extreme form, one tries to explain facts about a nucleus by considering the states of motion of a single nucleon in a field representing the combined effect of all the others. In a less extreme version one considers similarly the 'few' nucleons outside closed shells. This must be refined by including some effective residual interaction between those few nucleons: one tries to do this by considering them to be in states determined by the central field only, and



then introducing the residual interaction as a perturbation acting between these states.

As soon as the number of nucleons involved in the calculation becomes appreciably large, the number of linearly independent many-nucleon functions that can be formed from them grows very rapidly, so that one must have some criterion for choosing a few of these as zero-order eigenfunctions for the perturbation calculation. While it is becoming fairly practical to diagonalize large matrices numerically with electronic computers, it is still very desirable to limit the calculations to a few states so as to keep the whole procedure fairly transparent. Techniques derived from group theory have been of vital importance in this matter of classifying many-nucleon wave functions so as to provide such a criterion.

The many-nucleon wave functions of the shell model, which represent eigenstates of the whole nucleus, are linear combinations of products of single-nucleon functions; the latter are the eigenfunctions of the hamiltonian assumed to represent the effect of the central field. To set up a classification we can examine the behaviour of the many-nucleon wave functions under various suitably chosen groups of transformations, and then label each according to the way in which it transforms. In mathematical terms we consider the linear vector space of many-particle functions and we form suitable linear combinations so as to obtain basis functions of the irreducible representations of the groups involved. Thus we have a set of wave functions labelled according to the irreducible representation to which they belong, (and further labelled according to the particular row of the irreducible representation to which they belong). For a general treatment of this





and other applications of group theory to quantum mechanics the reader is referred to Hamermesh (H62).

In general a first classification will leave several functions bearing the same labels so that we will want to consider their behaviour under several different groups of transformations until we have a labelling scheme sufficiently distinctive for our purpose. In the preliminary stages this is a purely formal exercise. If the groups are judiciously chosen, however, the labelling scheme will reflect properties of the eigenstates, such as their order in the energy spectrum of the excited nucleus: some of the wave functions constructed to be base functions of selected irreducible representations will be good approximations to the actual wave functions of the nucleus, hence will be excellent starting-points for perturbation calculations and may even be worth talking about without further mixing. If so the classification will be said to be physically relevant.

Several group-theoretical classifications are known to be physically relevant. In particular the reader is assumed to be familiar with the angular momentum classification in which many-nucleon wave functions are assigned a label  $J$  according to their behaviour under the transformations of the rotation group.

In the case of the particular version of the LS-coupling shell model that assumes the central field to be of the harmonic oscillator type, with restoring force proportional to the distance from a fixed centre, the high degree of symmetry of this hamiltonian, leading to a degeneracy of the eigenfunctions, makes possible a classification according to the irreducible representations of the group called  $SU_3$  (this notation will be explained in Section 2.1). This classification is





found to be physically relevant, since some of the basis functions resemble wave functions previously calculated for actual nuclei.

What really makes this classification interesting however is that it is found to associate eigenstates of angular momentum into sets which correspond very closely to the so-called bands of the rotational model. Moreover a suitably chosen residual interaction leaves these  $SU_3$  functions unchanged when used in a perturbation calculation but assigns to them an energy spectrum resembling very closely that typical of the rotational model.

In a sense this is surprising because the collective models, including the rotational model, start from assumptions very different from those of the shell model. The collective approach, as its name implies, does not consider the individual nucleons but rather such phenomena as rotations of the whole nucleus considered more or less as a rigid body or a liquid drop, and surface oscillations of various types. In particular the rotational bands mentioned arise from rotations of the whole nucleus assumed to have cylindrical but not spherical symmetry. This mass or portions thereof may be in various states of rotational motion with respect to a coordinate system fixed to its own symmetry axis: these are the so-called intrinsic states. But each such intrinsic state, as the whole nucleus tumbles about in space, will appear to an observer as a whole set of different possible states of rotational motion with respect to a laboratory-fixed coordinate system. Each intrinsic state thus generates a rotational band, and the energy of the states within a band increases with angular momentum in a simple way involving a moment of inertia.

It is surprising that the shell model completed by the  $SU_3$



classification is able to reproduce these intrinsic states and rotational bands because the starting points of the two models are so different. But from a different point of view this can not be considered as surprising, for the independent particle model produces a set of states which is complete in a mathematical sense. Thus it must be possible to reproduce results of any other model so long as one is clever enough to choose suitable linear combinations from this complete set of states. The beauty of the  $SU_3$  result is that the choice that leads to rotational model results is not a contrived and obscure one devised for this purpose only, but stems instead from a very natural classification of the eigenfunctions of a very familiar potential (in fact one of the very few potentials which can be solved exactly in quantum mechanics). Harmonic oscillator wave functions, long before the  $SU_3$  classification was proposed, have been the starting point for virtually all shell model calculations.

## 1.2 History and Bibliography

Essentially all of the results described in the previous section are contained in the first two papers of J.P. Elliott on this subject, published in 1958. The first (E58) of these introduces the classification procedure, gives tables for the first few shells, and argues the physical relevance of the classification by comparing wave functions with those resulting from a previous shell model calculation for a few light nuclei. The second (E58a) establishes the significance of the classification with respect to the rotational model by introducing the



intrinsic states and deriving a quadrupole moment formula. These basic papers are readable and no attempt will be made to repeat their content beyond the purely qualitative outline already given, except for those topics to which further thought has been given.

The discussion following the presentation of Elliott's work to the Rehovoth Conference by Flowers (Fw58) brought out immediately what remains to this day the main limitation of  $SU_3$  theory: the fact that its success seems to depend significantly on the assumption of LS-coupling.

A third paper in this series, by Harvey and Elliott (Hv63), is to appear very soon. It presents a method for calculating matrix elements of a two-body interaction in  $SU_3$  wave functions and an application to  $Mg^{24}$ , with some discussion of other nuclei in the mass region  $16 < A < 40$ .

Work very similar to that of Harvey and Elliott has also been done by Banerjee, Levinson, Meshkov, and Pal, but it seems not to have been published. Some of it, also on  $Mg^{24}$ , was reported briefly to the Kingston Conference by Banerjee (Bj60).

Similarly, further applications by Elliott and by Levinson, Meshkov, Flamm, and Banerjee, to the extensive experimental results of Litherland, Kuehner, Gove, Clark, and Almqvist (Lh61), were discussed at the Manchester Conference (Mk61) and have just now been published (Fm63).

Some recent lecture notes by Elliott (E62) give more details than the publications on a few points.

Closely related developments have been given by several authors. In particular Moszkowski (Mz58), starting from considerations





emphasizing the importance of the quadrupole-quadrupole term in the residual interaction, gave an extensive study of the consequences of this for the relationships between the shell model and the collective model; he applied this first to a two-dimensional model, then to nuclei of the p shell. The classification with respect to  $SU_3$  may be looked upon as the key to extending this discussion to other shells as well. This quadrupole-quadrupole residual interaction of Moszkowski is the one which gives the  $SU_3$  functions an energy dependence analogous to rotational band spectra.

The  $SU_3$  classification was later obtained in a different way by Kretzschmar (K60,60b), with more emphasis on the group-theoretical manipulations and less on the physical relevance. The main advantage of the Kretzschmar formulation over others is a new method of eliminating spurious states, which will be explained later.

Another important series of papers in which the same classification is obtained independently, in yet another way, is that by Bargmann and Moshinsky (B60,61). The relations of this approach to the Elliott treatment will be discussed in a later chapter. In a more recent paper, Moshinsky (M62,62a) uses related techniques for treating the problem of  $n$  particles in a single shell of angular momentum  $\ell$ . He has also considered recently (M62b) some of the Wigner coefficients for the group  $SU_3$  with applications to the shell model and to other branches of physics.

A connexion between certain  $SU_3$  wave functions and the cluster model of Wildermuth and Kanellopoulos was pointed out by Bayman and Bohr (By58). More recent comments in a similar vein have been given by Inglis (In62).





R.S. Willey (Wi62) has studied in some detail the implications of the quadrupole-quadrupole residual interaction. D.S. Koltun (Ko61) has made intermediate coupling calculations in the  $p$  shell on the basis of the  $SU_3$  model.

The preceding outline was not meant to mention all relevant contributions, but only to indicate some of the interesting ones and to facilitate reference to them in the next section.

### 1.3 Organization of the thesis

The mathematical techniques which are applied or which could advantageously be applied to the problem of the classification of many-particle functions in terms of the symmetric and of the various unitary groups are not terribly profound or difficult, but they are known to a very limited number of physicists and mathematicians. No suitable account of them seems to exist, presenting all the techniques needed in a way that facilitates application; and the vocabulary and notation are very confused. As a result even those physicists who do use these techniques tend to avoid explaining their application, and often avoid mentioning them altogether.

It was therefore thought desirable to give a utilitarian presentation of this subject in the present thesis, including detailed illustrative examples. Chapter 2 presents all the purely mathematical results used in this work, and Chapter 3 is an account of their application to many-particle functions. The mathematical presentation differs in various ways from such previous presentations as could be



found: the Young Diagrams themselves, because they can label the irreducible representations of various groups, are considered as mathematical objects in their own right and various operations are defined in terms of them.

The use of Young Diagrams both for symmetric and for linear groups is introduced in Section 2.1, and various types of Kronecker products of representations are introduced in Section 2.2, along with the related concept of Foulkes operators. The third section introduces the concept of plethysm and applies it to linear groups (ordinary plethysm). Section 2.4 discusses plethysm for symmetric groups (inner plethysm) and ends with some remarks on the numerical evaluation of various types of products and plethysms.

Section 3.1 stresses various points which I have found helpful in understanding the important relationship between the symmetric-group and the linear-group classifications of many-particle functions. Section 3.2 applies these classifications to the many-nucleon eigenfunctions of the shell model in LS coupling. The third section of this chapter explains the use of plethysm techniques for evaluating the subductions which arise when classifications with respect to two different groups are applied simultaneously. And a final section develops a formalism which makes it possible to extend the same classification procedures to excited configurations as well as ground configurations.

The important relationships between the supermultiplet model developed by Wigner and others in 1937 and the  $SU_3$  model which is the main concern of this thesis are considered in some detail. For one thing the supermultiplet model is a prefiguration of the  $SU_3$  model



in that it gives the same interesting results in the special case of the  $p$  shell. Moreover the  $SU_3$  classification for any shell is introduced as a subclassification of the supermultiplet classification; and when the  $SU_3$  model is used to choose states which are likely to be low in the energy spectrum of a nucleus, the states chosen are those that are simultaneously favoured by both models, although no one has bothered to set up a model formally combining the assumptions of both. To understand these interrelationships it is necessary to learn in what sense the supermultiplet classification, introduced historically in terms of the symmetric group, is at the same time a classification in terms of a unitary group  $SU_s$ . This is part of the reason for the detailed considerations of Chapter 3 and for Section 4.1, which introduces the supermultiplet model for the  $p$  shell and those aspects of it that remain valid in higher shells also. As for Section 4.2, it applies the supermultiplet model to excited configurations using the techniques given in Section 3.4; part of the motivation here is to prepare the way for the application of the  $SU_3$  model to excited configurations in the next chapter.

Chapter 5 does not attempt to paraphrase the content of Elliott's publications on the  $SU_3$  classification; this has already been done qualitatively in Section 1.1 and it would not be useful to expand it. But we do describe the parallel between the  $SU_3$  and supermultiplet classifications in a way which shows why all the developments described in Chapter 3 are readily adapted to the  $SU_3$  classification.

The second section of Chapter 5 introduces the  $SU_3$  model, and the third illustrates it by presenting a detailed application to the experimental energy spectrum of  $Ne^{20}$ , with special attention to the





excited configurations. In particular the techniques developed so far in the thesis are used to discuss the position of the lowest excited-configuration state in the energy spectrum. Some remarks are made in Section 5.4 on the uses and limitations of  $SU_3$  theory, including the reasons why such applications of the  $SU_3$  model as presented in Section 5.3 are entirely untypical. Some attention is also given to the problem of extending the  $SU_3$  classification or something analogous to it to nuclei too heavy for the LS-coupling assumptions to be reasonable.

Whereas Chapter 5 used almost exclusively the formalism and language of Elliott's original publications, Chapter 6 concerns two of the several series of publications in which different authors have obtained either similar or identical results through different formalisms. Such situations are far from uncommon in theoretical physics but in this particular section of the subject it seems to be more difficult than usual to pin down the precise relations between some of the formalisms. The work of Kretzschmar is examined (in Section 6.1) for two reasons: (a) it is similar in intent to part of the present thesis, stressing the group-theoretical techniques and the relationships between various possible treatments, and (b) it presents a useful technique for dealing with spurious states (using inner plethysm), which we apply to the first excited-configuration state of  $Ne^{20}$  already discussed in Section 5.3. Finally in Section 6.2 we use some developments given by Kretzschmar to establish the relationship between the formalism of Bargmann and Moshinsky and that of Elliott, and we present a calculation carried out in physical terms in the transformation-bracket formalism of Moshinsky, which also helps to understand the relations between different formalisms.

A concluding chapter includes among other things a list of some topics on which further work would be desirable.





## Chapter 2      YOUNG DIAGRAM MANIPULATIONS

### 2.1    Young Diagrams, Symmetric and Linear Groups

All the mathematical results used in the thesis are assembled in this chapter. The theory of representations (by homogeneous linear transformations) of the symmetric group (the group of permutations of  $k$  objects) and of the various linear groups (whose group elements are matrices subject to various restrictions) is well established in its essentials, but research on some aspects of it is still being published. Powerful techniques emerging from this research have been applied by several authors (H62) to the physical problem of the classification of quantum-mechanical states of many-particle systems, such as the nucleus; some of them will be reviewed and used in this thesis. The key to this branch of mathematics is a fundamental and far-reaching correspondence, sometimes called duality, between the symmetric group on one hand and the linear groups on the other. Some aspects of this duality require considerable mathematical sophistication. Neglecting these, the present chapter provides a simplified utilitarian presentation of the essential points and of the techniques needed in this physical problem, without proofs or mathematical motivation. The physical motivation will be given in the next chapter. There is need for a more elaborate work along the same lines, extending to some allied techniques not presented here. For the mathematical literature on this subject is difficult to approach for the physicist: different points of view (i.e. symmetric vs linear) naturally lead authors to different and often conflicting terminology and notation.



Because of this and of the fact that the literature is plagued by misprints, it is unsafe to use numerical results quoted by others without the sort of dimensional checks of various kinds which are emphasized below.

Opportunities for mathematical insight will be sacrificed to make the use of the techniques as unambiguous as possible. This presentation is based upon the concise appendix given by Kretzschmar (K60), but amplifies it considerably.

Lomont's book (Lo59) and a recent text by Hamermesh (H62) on group theory in quantum mechanics will serve as general references for standard concepts not defined below. An organized mathematical presentation is given in Robinson's textbook (R61).

Following Melvin (Mv 56), I use the word rep for irreducible representation over the field of complex numbers: thus a rep is a homomorphic mapping of group elements onto complex non-singular square matrices, and its underlying linear vector space, or carrier space, contains no invariant subspace.

Two ordered sets of matrices  $\{D_i\}$  and  $\{D_i'\}$  are equivalent if one can be obtained from the other by a single similarity transformation  $S\{D_i\}S^{-1} = \{D_i'\}$ . Representations that are equivalent in this sense form a representation class; in particular, equivalent reps form a rep class.

A YD (for Young Diagram)  $[\lambda] = [\lambda_1, \lambda_2, \dots, \lambda_i, \dots]$  is defined as a non-increasing sequence of non-negative integers  $\lambda_i$ ,

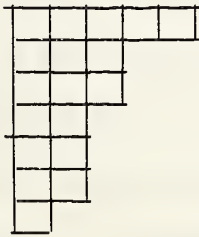
$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_i \geq \dots \geq 0, \quad 2.1$$

and pictured as an array of boxes (or squares, nodes, dots, x's, etc.), the  $i$ 'th row containing  $\lambda_i$  boxes. The total number of boxes is  $\#[\lambda] = \sum_i \lambda_i$ , so that  $[\lambda]$  is a partition of the integer  $\#[\lambda]$ . Two YD's that differ only in the number of zeros they contain are considered as the same



YD; hence the number of rows of a YD is indeterminate. (In fact the zeros are usually omitted in the symbol  $[\lambda]$  except in the case of the useful symbol  $[0]$ , for the YD with  $\#[0] = 0$ .) But the number of non-vanishing rows of a YD, i.e. the number of positive integers  $\lambda_i$  in  $[\lambda]$  is well defined and will be denoted  $\text{NR}[\lambda]$ . It is sometimes called 'the number of rows'. A latin letter in square brackets usually denotes a special YD of, say,  $m = \#[m]$  boxes in a single row:  $\text{NR}[m] = 1$ , and  $\text{NR}[0] = 0$ .

Commas are often omitted in the symbol  $[\lambda]$ , and for conciseness an upper index on a  $\lambda_i$  indicates the number of times that particular integer is repeated. Thus  $[53^22^31] = [5, 3, 3, 2, 2, 2, 1] = [5, 3, 3, 2, 2, 2, 1, 0, 0, \dots]$

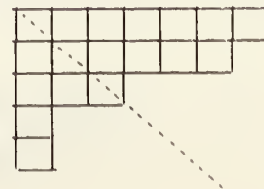
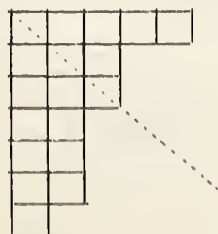


is pictured as shown.  $\#[53^22^31] = 18$ , and  $\text{NR}[53^22^31] = 7$ .

Another useful notation for  $[\lambda]$  is  $[c\mu]$ , where  $[\mu] = [\lambda_2, \lambda_3, \dots, \lambda_i, \dots]$ , and the  $c$  reminds the reader that an indeterminate  $\lambda_1$  is omitted in the symbol. This

implies that  $\lambda_1 = \#[\lambda] - \#[\mu]$  is specified from the context. Thus  $[c3^22^31] = [53^22^31]$  provided one specifies that  $\#[\lambda] = 18$ . The same symbol  $[c3^22^31]$  will designate  $[63^22^31]$  if  $\#[\lambda] = 19$ , or  $[33^22^31]$  if  $\#[\lambda] = 16$ , and so on.

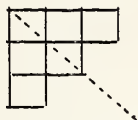
The YD  $[\tilde{\lambda}]$  obtained from  $[\lambda]$  by reflecting the diagram in the main diagonal (upper left to lower right), so that rows become columns and columns rows, is said to be associate to  $[\lambda]$ ; (other authors use conjugate or adjoint). For instance the associate of  $[53^22^31]$  is  $[7631^2]$ .







Obviously association is involutory, i.e.  $[\tilde{\lambda}] = [\lambda]$ , and  $\#[\tilde{\lambda}] = \#[\lambda]$ .  $[\lambda]$  is said to be self-associate if  $[\tilde{\lambda}] = [\lambda]$ : e.g.  $[321]$  is self-associate.



Of great interest is the symmetric group  $S_k$  whose elements are the  $k!$  permutations of  $k$  objects. Its rep classes are in one-one correspondence with the YD's of  $k$  boxes:  $\#[\lambda] = k$ . Thus the number of rep classes of  $S_k$  is  $p(k)$ , the number of partitions of  $k$ . Numerical tables of  $p(k)$  are given by Lomont (Lo59) for selected  $k$  up to  $p(200) = 3972999029388$ , and by Osima (O52) for all  $k$  up to  $p(40) = 37388$ ;  $p(10)$  is 42, and  $p(20)$  is 627.

It is useful to have a standard order for systematic listing of all  $[\lambda]$  with  $\#[\lambda] = k$ . One such order puts  $[\lambda]$  before  $[\lambda']$  if  $\lambda_1 > \lambda'_1$ ; or if  $\lambda_1 = \lambda'_1$  and  $\lambda_2 > \lambda'_2$ ; or if  $\lambda_1 = \lambda'_1$ ,  $\lambda_2 = \lambda'_2$ ,  $\lambda_3 > \lambda'_3$ ; etc. A glance at Table 2-1 will make this clear.

The dimension (or degree) of a representation (or representation class) is the number of rows or of columns in the square matrices which constitute the representation. The dimension of the rep class of  $S_k$  labelled by the YD  $[\lambda]$  is thus a positive integer denoted by  $\dim_{S_k}[\lambda]$ . Note that it would be ambiguous to designate this number simply by  $\dim[\lambda]$ , because the same YD's  $[\lambda]$  will also label the rep classes of various linear groups, and those reps have dimension different from  $\dim_{S_k}[\lambda]$ .

Formulas for  $\dim_{S_k}[\lambda]$  have been rederived recently by Robinson (R56). If  $h_{ij} = (\lambda_i - i) + (\lambda'_j - j) + 1$ , where  $\lambda'_j$  is the length of the  $j$ 'th column of  $[\lambda]$ , then

$$\dim_{S_k}[\lambda] = \frac{k! \prod_{i < j} (h_{i1} - h_{j1})}{\prod_i (h_{i1}!)} . \quad 2.2$$





TABLE 2-1  $\dim G[\lambda]$  and  $\Xi$ 

$k$	$[\lambda]$	$\dim S_k[\lambda]$	$\Xi$	$\dim U_n[\lambda]$				
				$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
1	$[1]$	1	0	2	3	4	5	6
2	$[2]$	1	1	3	6	10	15	21
	$[1^2]$	1	-1	1	3	6	10	15
3	$[3]$	1	3	4	10	20	35	56
	$[21]$	2	0	2	8	20	40	70
	$[1^3]$	1	-3		1	4	10	20
4	$[4]$	1	6	5	15	35	70	126
	$[31]$	3	2	3	15	45	105	210
	$[2^2]$	2	0	1	6	20	50	105
	$[21^2]$	3	-2		3	15	45	105
	$[1^4]$	1	-6			1	5	15
5	$[5]$	1		6	21	56	126	252
	$[41]$	4	5	4	24	84	224	504
	$[32]$	5	2	2	15	60	175	420
	$[31^2]$	6	0		6	36	126	336
	$[2^21]$	5	-2		3	20	75	210
	$[21^3]$	4	-5			4	24	84
	$[1^5]$	1	-10				1	6
6	$[6]$	1		7	28	84	210	462
	$[51]$	5		5	35	140	420	1050
	$[42]$	9	5	3	27	126	420	1134
	$[41^2]$	10	3		10	70	280	840
	$[3^2]$	5	3	1	10	50	175	490
	$[321]$	16	0		8	64	280	896
	$[31^3]$	10	-3			10	70	280
	$[2^3]$	5	-3		1	10	50	175
	$[2^21^2]$	9	-5			6	45	189
	$[21^4]$	5	-9				5	35
	$[1^6]$	1	-15					1
7	$[7]$	1		8	36	120	330	792
	$[61]$	6		6	48	216	720	1980
	$[52]$	14		4	42	224	840	2520
	$[51^2]$	15			15	120	540	1800
	$[43]$	14	6	2	24	140	560	1764
	$[421]$	35	3		15	140	700	2520
	$[41^3]$	20	0			20	160	720
	$[3^21]$	21	1		6	60	315	1176
	$[32^2]$	21	-1		3	36	210	840
	$[321^2]$	35	-3			20	175	840
	$[31^4]$	15	-7				15	120
	$[2^31]$	14	-6			4	40	210
	$[2^21^3]$	14	-9				10	84
	$[21^5]$	6	-14					6
	$[1^7]$	1	-21					



Pictorially, if we define a Young Tableau as an arrangement of the integers  $1, 2, \dots, k$  in the  $k$  boxes of  $[\lambda]$  such that no integer appears directly above or directly to the left of any integer smaller than itself, then  $\dim S_k[\lambda]$  is given by the number of such arrangements consistent with the YD  $[\lambda]$ . Thus  $\dim S_5[32] = 5$ .

1	2	3
4	5	

1	2	5
3	4	

1	2	4
3	5	

1	3	5
2	4	

1	3	4
2	5	

One sees readily that  $\dim S_k[\tilde{\lambda}] = \dim S_k[\lambda]$ ;  $\dim S_k[k] = \dim S_k[1^k] = 1$ ; and  $\dim S_k[c1] = \dim S_k[21^{k-2}] = k - 1$ . Table 2-1 gives  $\dim S_k[\lambda]$  for  $k \leq 7$ .

From the general theory  $\dim S_k[\lambda]$  is the character of the identity class in the rep class  $[\lambda]$  of  $S_k$ . Consider also the character  $\chi(T)$  of the class consisting of a single transposition. Then the integer (Hu37)

$$\Xi = \frac{k(k-1)\chi(T)}{\dim S_k[\lambda]} = \frac{1}{2} \{ \lambda_1(\lambda_1 - 1) + \lambda_2(\lambda_2 - 3) + \lambda_3(\lambda_3 - 5) + \dots \}, \quad \underline{2.3}$$

is a quantity of physical interest, as discussed in Chapter 4; it also appears in Table 2-1, for those  $[\lambda]$  for which it is physically relevant.

Consider now the linear groups. The full linear group in  $n$  dimensions  $GL_n$  has as its elements the non-singular  $n \times n$  matrices of complex numbers. Its subgroup  $SL_n$ , where  $S$  means special-unimodular, is restricted to those matrices having determinant  $+1$ ; (this is often called unimodular but special-unimodular is preferable because unimodular is then available for the case of determinants having absolute value 1). Another interesting subgroup of  $GL_n$  is  $GL_n^1$ , in which case the matrices must be real; if they are not only real but special-unimodular as well, we have  $SL_n^1$ . More interesting still for quantum mechanics are the subgroup  $U_n$ , of  $n \times n$  unitary matrices ( $UU^* = 1$ , where  $*$  means



hermitian adjoint) and its special-unimodular subgroup  $SU_n$ , in which the matrices must be both unitary and of determinant +1. The reps of all the groups mentioned in this paragraph are very closely related, as described below.

The concept of subduction will occur frequently throughout this thesis. Let  $\Gamma$  be a representation of a finite or compact group  $G$ . Consider now those matrices of  $\Gamma$  which represent the elements of a subgroup  $H \subset G$ . They form a representation  $\Gamma'$  of  $H$ , in general reducible, which is said to be obtained from  $\Gamma$  by the subduction  $G \rightarrow H$ .  $\Gamma'$  can then be expressed as a direct sum of reps of  $H$ .

We will be interested only in those reps of the linear groups for which the matrix elements of the matrices of the rep are homogeneous polynomials of degree  $r$  in the matrix elements of the group elements represented; the basis functions of such reps are irreducible tensors of rank  $r$  with respect to the linear group and will be discussed explicitly in the next chapter. Such irreducible tensors have symmetry type  $[\lambda]$ , where  $\#[\lambda] = r$ , under permutations of their  $r$  indices. Thus the rep classes of  $GL_n$  are in one-one correspondence with certain YD's. As discussed in Hamermesh (H62) and elsewhere, each  $[\lambda]$  such that  $NR[\lambda] \leq n$  labels a rep class of  $GL_n$ .

For the subgroups mentioned above one can show (H62) that the reps  $[\lambda]$  of  $GL_n$  remain irreducible under the subductions  $GL_n \rightarrow GL'_n$  and  $GL_n \rightarrow U_n$ , and the rep classes of  $GL'_n$  or  $U_n$  are still labelled by the same YD's. Under the subductions to special-unimodular subgroups however, such as  $GL_n \rightarrow SL_n$  or  $U_n \rightarrow SU_n$ , it is still true that the reps  $[\lambda]$  remain irreducible, but some rep classes now coalesce. All YD's obtained from a given  $[\lambda]$  by adding or removing any number of complete



columns of  $n$  boxes become equivalent. Moreover the reps labelled by  $[\lambda]$  become equivalent to those labelled by the complement  $[\lambda^n]$  of  $[\lambda]$ , which is defined by  $\lambda_i^n = \lambda_1 - \lambda_{n-i+1}$ . Pictorially the complement is obtained by inverting that portion of the rectangle  $\lambda_1$  boxes long and  $n$  boxes high which is not occupied by  $[\lambda]$ . Thus the complement of  $[421]$  for  $n=3$  is  $[32]$ , while the complement of  $[421]$  for  $n=4$  is  $[432]$ .

		x	x
	x	x	x

		x	x
	x	x	x
x	x	x	x

For the group  $SU_3$ ,  $[1]$  is equivalent to  $[1^2]$  by the complement rule, and to  $[21^2]$  by the complete columns rule.

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Note that  $[1]$  is not equivalent to  $[21]$  for  $SU_3$  because a complete column has 3 boxes; however  $[1]$  is equivalent to  $[21]$  for  $SU_2$ .

Because of the complete column rule one could say that the rep classes of  $SU_n$  are given by the YD's obeying  $NR[\lambda] \leq n-1$ , complements being identified. There are advantages however to thinking in terms of the rule  $NR[\lambda] \leq n$ , and remembering the equivalence rules separately.

It is obvious from the nature of the subduction process that  $\dim GL_n[\lambda] = \dim SL_n[\lambda] = \dim U_n[\lambda] = \dim SU_n[\lambda] = \dots$ . The symbol  $\dim U_n[\lambda]$  will be used for all because it is shorter. The formula

$$\dim U_n[\lambda] = \prod_{1 \leq i < j \leq n} \left\{ \frac{\lambda_i - \lambda_j + j - i}{j - i} \right\} \quad \underline{2.4}$$

follows from the general character formula given by Weyl (We39).





Interesting special cases of this are

$$\dim U_n[1] = n, \quad \underline{2.5}$$

$$\dim U_n[2] = \frac{1}{2}n(n+1), \quad \underline{2.6}$$

$$\dim U_n[1^2] = \frac{1}{2}n(n-1), \quad \underline{2.7}$$

$$\dim U_n[1^k] = \binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \underline{2.8}$$

$$\dim U_n[m] = \dim U_{m+1}[n-1], \quad \underline{2.9}$$

$$\dim U_n[\tilde{\lambda}] = \dim U_{n-k+1}[\lambda], \quad \underline{2.10}$$

$$\dim U_2[\lambda] = \lambda_1 - \lambda_2 + 1, \quad \underline{2.11}$$

$$\dim U_3[\lambda] = \frac{1}{2}(\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_1 - \lambda_3 + 2), \quad \underline{2.12}$$

where  $k = \#[\lambda]$ , and  $[\lambda]$  precedes  $[\tilde{\lambda}]$  in the standard order previously mentioned. Table 2-1 gives  $\dim U_n[\lambda]$  for  $\#[\lambda] \leq 7$  and  $n \leq 6$ . Blank spaces appear if  $NR[\lambda] > n$ ; a useful convention is to set  $\dim U_n[\lambda] = 0$  in these cases.

To end this section let us establish a link with something familiar. The rep classes of  $SU_2$ , which is homomorphic to the rotation group  $R_3$ , are well known: they are labelled by a single number  $j = 0, \frac{1}{2}, 1, 3/2, \dots$ , and their dimension is  $2j+1$ . On the other hand the YD scheme would identify them with the  $[\lambda]$  such that  $NR[\lambda] \leq 2$ , but with complete columns of two boxes considered irrelevant, so that only the quantity  $\lambda_1 - \lambda_2$  is significant. The relationship between the two labelling schemes is fixed by comparing the dimension formulas:

$$\dim U_2[\lambda] = \lambda_1 - \lambda_2 + 1 = 2j+1,$$

therefore  $j = \frac{1}{2}(\lambda_1 - \lambda_2).$  2.13

Thus the rep class  $j=0$  is described by any one of the YD's  $[0]$ ,  $[1^2]$ ,  $[2^2]$ ,  $[3^2]$ , etc. with  $\dim U_2[\lambda] = 1$ ;  $j=\frac{1}{2}$  by  $[1]$ ,  $[21]$ ,  $[32]$ ,  $[43]$ , etc. with  $\dim U_2[\lambda] = 2$ ;  $j=1$  by  $[2]$ ,  $[31]$ ,  $[42]$ ,  $[53]$ , etc. with  $\dim U_2[\lambda]=3$ ;



and so on.

For completeness note that the reps of other subgroups of  $GL_n$ , such as the symplectic and the orthogonal groups, are also related to this theory, and developments similar to those of the next few sections can also be given for them (H62, L58); they will not occur in this thesis, however.

## 2.2 Kronecker Products, Inner and Outer

If  $D$  and  $D'$  are two matrices of dimension  $d$  and  $d'$  respectively, let  $D \times D'$  denote their Kronecker product, of dimension  $dd'$ . Lomont (Lo59) distinguishes carefully between the inner Kronecker product and the outer Kronecker product of two ordered sets of matrices  $\Gamma = \{D_1, D_2, \dots, D_N\}$  of dimension  $d$ , and  $\Gamma' = \{D'_1, D'_2, \dots, D'_{N'}\}$  of dimension  $d'$ . The inner Kronecker product can be defined only if  $N' = N$ : it is the set of  $N$  matrices

$$\Gamma \dot{\times} \Gamma' = \{D_1 \times D'_1, D_2 \times D'_2, \dots, D_N \times D'_N\}. \quad 2.14$$

The outer Kronecker product of  $\Gamma$  and  $\Gamma'$ , which does not require that  $N' = N$ , is the set of all  $NN'$  matrices  $D_i \times D'_j$  where  $i = 1, 2, \dots, N$ , and  $j$  independently runs over  $1, 2, \dots, N'$ :

$$\Gamma \ddot{\times} \Gamma' = \{D_1 \times D'_1, D_1 \times D'_2, \dots, D_1 \times D'_{N'}, D_2 \times D'_1, \dots, D_N \times D'_{N'}\}. \quad 2.15$$

Let us apply these concepts now to the reps of  $S_k$ . Consider two reps  $\Gamma_{[\lambda]}$  and  $\Gamma_{[\lambda']}$  of the same group  $S_k$ . Their inner Kronecker product as defined above is then also a representation of  $S_k$  but in general reducible. Its reduction, i.e. its expression as a direct sum of reps of  $S_k$  designated by  $\Gamma_{[\lambda'']}$ ,



$$\Gamma_{[\lambda]} \times \Gamma_{[\lambda']} = \sum_{[\lambda'']} g_{\lambda \lambda' \lambda''} \Gamma_{[\lambda'']}, \quad 2.16$$

is the so-called Clebsch-Gordan series for  $S_k$ . The sum is over all YD's of  $k$  boxes, and the coefficients  $g_{\lambda \lambda' \lambda''}$  (which are not the Clebsch-Gordan coefficients of this group) are non-negative integers some of which are greater than one (i.e.  $S_k$  is not multiplicity-free for  $k \geq 5$ ).

Because the YD's, as we have seen, are linked to several groups, it is useful to consider them as mathematical objects in their own right and to define operations involving them. We define first the sum of YD's to correspond to the direct sum of reps:  $[\alpha] + [\beta]$  labels the representation formed by taking the direct sum of the reps labelled  $[\alpha]$  and  $[\beta]$ . For reasons which become clear when we examine the actual basis functions in the next chapter, the statement remains true whether we mean the reps of symmetric or of linear groups.

We then define the in-product of two YD's by

$$[\lambda] \cdot [\lambda'] = \sum_{[\lambda'']} g_{\lambda \lambda' \lambda''} [\lambda''], \quad 2.17$$

with the same coefficients  $g_{\lambda \lambda' \lambda''}$  as above. A procedure for the evaluation of such in-products exists (H62) but will not be given here.

Murnaghan (Mu38) gives explicit tables for  $k \leq 8$ , and 58 semi-general formulas of the following type, involving the  $[c_\mu]$  notation previously defined (which leaves  $\lambda_1$  to be determined from the context):

$$[c1] \cdot [c1] = [c0] + [c1] + [c2] + [c1^2]; \quad 2.18$$

$$[c1] \cdot [c2] = [c1] + [c2] + [c1^2] + [c3] + [c21]; \quad 2.19$$

$$[c1] \cdot [c1^2] = [c1] + [c2] + [c1^2] + [c21] + [c1^3]; \quad 2.20$$

$$[c2] \cdot [c2] = [c0] + [c1] + 2[c2] + [c1^2] + [c3] + 2[c21] + [c1^3] + [c4] + [31] + [c2^2]. \quad 2.21$$

As an example, when the third of these is used for  $S_5$ , it reads





$$[41] \cdot [31^2] = [41] + [32] + [31^2] + [2^2 1] + [21^3];$$

for the group  $S_6$  the same formula reads

$$[51] \cdot [41^2] = [51] + [42] + [41^2] + [321] + [31^3].$$

In that sense the in-product formulas are independent of  $k$ .

Note however that if we apply the same formula to  $S_4$ , we get

$$[31] \cdot [21^2] = [31] + [2^2] + [21^2] + [121] + [1^4];$$

the term  $[121]$  appears which violates our requirement that the  $\lambda_i$  be a non-increasing sequence. Whenever such generalized YD's occur, in which the  $\lambda_i$  may be any integers (positive, negative, or zero) and the order may be disarranged, (i.e. may not obey  $\lambda_1 \lambda_2 \dots 0$ ), they are to be handled according to the following modification rules.

Rule 1:  $[\dots, \lambda_i, \lambda_{i+1}, \dots] = -[\dots, \lambda_{i+1} - 1, \lambda_i + 1, \dots]$  if  $\lambda_{i+1} > \lambda_i + 1$ ;

Rule 2:  $[\lambda_1, \lambda_2, \dots, \lambda_r] = 0$  if  $\lambda_r < 0$ , where  $\lambda_r$  is the last non-vanishing  $\lambda_i$ .

Rule 3:  $[\lambda_1, \lambda_2, \dots, \lambda_r] = 0$  if  $\lambda_{i+1} = \lambda_i + 1$  for any  $i$ .

Given a disarranged YD, Rule 1 is to be applied as often as necessary, until either a generalized YD is obtained that vanishes by Rule 2 or Rule 3, or an acceptable YD (in correct order, and without negative integers) results. If at this stage this YD bears a minus sign (from Rule 1 applied an odd number of times), the same YD necessarily arises with a plus sign from another term in the formula, and the two will simply cancel; no minus signs should remain at the end.

In the example above,  $[121]$  immediately vanishes by Rule 3. As a more elaborate example, the last semi-general formula quoted, applied to  $S_4$ , gives as explained below,

$$\begin{aligned} [2^2] \cdot [2^2] &= [4] + [31] + 2[2^2] + [21^2] + [13] + 2[121] + [1^4] + [04] + [031] + [02^2] \\ &= [4] + [31] + 2[2^2] + [21^2] - [2^2] + [1^4] - [31] - [21^2] - [1^2 2] \end{aligned}$$



$$= [4] + [2^2] + [1^4].$$

Here  $[13]$  became  $-[2^2]$  by Rule 1, cancelling one of the two  $[2^2]$  originally present;  $2[121]$  vanished immediately by Rule 3;  $[04]$  became  $-[31]$  and cancelled the  $+[31]$ ;  $[031]$  similarly became  $-[21^2]$  and cancelled; and  $[02^2]$  became  $-[1^2]$  which vanished by Rule 3.

A dimensional check is obviously desirable. Because the definition of in-product is based on Kronecker products of  $S_k$  we have simply

$$\dim S_k[\lambda] \dim S_k[\lambda'] = \sum_{[\lambda'']} g_{\lambda \lambda' \lambda''} \dim S_k[\lambda'']. \quad \underline{2.22}$$

Reading off dimensions from Table 2-1 the checks for the examples already given are easily carried out:

$$[41] \cdot [31^2] = [41] + [32] + [31^2] + [2^2 1] + [21^3],$$

$$4 \times 6 = 4 + 5 + 6 + 5 + 4;$$

$$[51] \cdot [41^2] = [51] + [42] + [41^2] + [321] + [31^3],$$

$$5 \times 10 = 5 + 9 + 10 + 16 + 10;$$

$$[31] \cdot [21^2] = [31] + [2^2] + [21^2] + [1^4],$$

$$3 \times 3 = 3 + 2 + 3 + 1;$$

$$[2^2] \cdot [2^2] = [4] + [2^2] + [1^4],$$

$$2 \times 2 = 1 + 2 + 1.$$

Note that the first three of these checks bear upon the same semi-general formula 2.20. Since such formulas apply for all  $k$ , a suspected misprint can quickly be tracked down by applying checks for several values of  $k$ .

Important special cases of the in-product coefficients  $g_{\lambda \lambda' \lambda''}$  are  $g_{\lambda \lambda' [k]} = \delta_{\lambda \lambda'}$ , and  $g_{\lambda \lambda' [1^k]} = \delta_{\lambda' \lambda}$ , where the Kronecker deltas mean that the rep  $[k]$  occurs in an in-product if and only if the two factors are the same YD, while the rep  $[1^k]$  occurs if and only if the two factors are associate YD's.



This completes the discussion of the in-product of YD's. We have applied the notion of inner Kronecker product to two reps of the same symmetric group  $S_k$  and defined on this basis the in-product of two YD's.

Let us now consider the outer Kronecker product of a rep  $\Gamma_{[\lambda]}^k$  of  $S_k$  by a rep  $\Gamma_{[\lambda']}^{k'}$  of  $S_{k'}$ . The set of matrices obtained is a rep of a group called the direct product  $S_k \times S_{k'}$  of the two groups  $S_k$  and  $S_{k'}$ . This group  $S_k \times S_{k'}$  is a subgroup of  $S_{k+k'}$ , its elements being those permutations of the  $k+k'$  objects which operate only within each of the subsets of  $k$  and  $k'$  objects respectively, but make no interchanges between the subsets: each element of  $S_k \times S_{k'}$  consists of one element of  $S_k$  and one element of  $S_{k'}$ .

We would now like to consider this rep of  $S_k \times S_{k'}$  as a representation of  $S_{k+k'}$ . Since  $S_k \times S_{k'} \subset S_{k+k'}$ , that is more or less the converse of the process of subduction previously discussed. But here of course  $\Gamma_{[\lambda]}^k \otimes \Gamma_{[\lambda']}^{k'}$  is not as it stands a representation of  $S_{k+k'}$ , for we lack matrices to represent those elements of  $S_{k+k'}$  not contained in  $S_k \times S_{k'}$  (i.e. those permutations which violate the segregation of the subsets  $k$  and  $k'$ ). A representation of  $S_{k+k'}$  called the induced representation and described below can nevertheless be constructed by a standard procedure, embodying  $\Gamma_{[\lambda]}^k \otimes \Gamma_{[\lambda']}^{k'}$  into larger matrices.

The very powerful concept of induced representation has been described as follows by Mackey (Ma51). Let  $G$  be a finite group,  $H$  a subgroup of  $G$ , and  $L$  (prescribing an  $L_h$  for each  $h \in H$ ) a representation of  $H$  by non-singular linear transformations  $L_h$  in a vector space  $V$  which is finite-dimensional over a field  $F$ . Consider functions  $f$ , with elements of  $G$  as independent variable and elements of  $V$  as dependent variable.  $L_h f(g)$  is then also an element of  $V$ , obtained by applying





the transformation  $L_h$  to  $f(g)$ . The functions  $f$  satisfying

$$f(hg) = L_h f(g)$$

for all  $g \in G$  and all  $h \in H$  form a vector space  $W$  over  $F$ , such that if  $f$  is in  $W$ , then so is the function  $f'$  defined by

$$f'(g) = f(gg')$$

for a fixed  $g'$  and all  $g$  in  $G$ . If we define  $U_s^L$  for all  $s \in G$  by the equation

$$\{U_s^L(f)\}(g) = f(gg'),$$

it follows that  $U_s^L$  transforms elements of  $W$  into elements of  $W$ . Then  $U^L$ , which maps all elements  $s \in G$  into non-singular linear transformations in the vector space  $W$  of functions  $f$ , is a representation of  $G$ , called the (imprimitive) representation 'induced' by the representation  $L$  of the subgroup  $H$ .

In the present application the group  $G$  is  $S_{k+k'}$  and the subgroup  $H$  is  $S_k \times S_{k'}$ . The representation  $L$  is the outer Kronecker product  $\Gamma_{[\lambda]}^k \otimes \Gamma_{[\lambda']}^{k'}$  and we denote the representation of  $S_{k+k'}$  induced by  $L$  as  $\{\Gamma_{[\lambda]}^k \otimes \Gamma_{[\lambda']}^{k'}\}^\uparrow$ . In general the induced representation is reducible and we consider its reduction into a direct sum of reps of  $S_{k+k'}$  labelled by YD's  $[\lambda'']$ :

$$\{\Gamma_{[\lambda]}^k \otimes \Gamma_{[\lambda']}^{k'}\}^\uparrow = \sum_{[\lambda'']} t_{\lambda \lambda' \lambda''} \Gamma_{[\lambda'']}^{k+k'} \quad \underline{2.23}$$

Again we define an operation on YD's, called the out-product of YD's, by

$$[\lambda] \circ [\lambda'] = \sum_{[\lambda'']} t_{\lambda \lambda' \lambda''} [\lambda''], \quad \underline{2.24}$$

with the same non-negative integer coefficients  $t_{\lambda \lambda' \lambda''}$  as above. In this case  $\#[\lambda] = k$  and  $\#[\lambda'] = k'$  may be equal but need not be, and  $\#[\lambda'']$  is  $k + k'$ . Explicit tables are given by Murnaghan (Mu37) for  $\#[\lambda''] \leq 10$ .

A convenient graphical procedure for evaluating the out-products is given by Littlewood (L50, H62). The procedure is awkward to describe









$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \circ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & & 2 & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & & & \\ \hline 2 & & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & & 2 \\ \hline 1 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & & \\ \hline 1 & 2 & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & & \\ \hline 1 & & \\ \hline 2 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & & \\ \hline 1 & 1 & \\ \hline 2 & & \\ \hline \end{array} ;$$

$$[2^2] \circ [21] = [43] + [421] + [3^21] + [32^2] + [321^2] + [2^31].$$

A dimensional check for out-products is

$$\binom{k+k'}{k} \dim S_k[\lambda] \dim S_{k'}[\lambda'] = \sum_{[\lambda'']} \dim S_{k+k'}[\lambda''], \quad \underline{2.25}$$

where the binomial coefficient arises from the construction of the induced representation. For the last example above this gives (using Table 2.1)

$$\frac{7!}{4! 3!} 2 \times 2 = 140 = 14 + 35 + 21 + 21 + 35 + 14.$$

We have defined the out-product of YD's to correspond to the outer Kronecker product of reps of two symmetric groups. But it can be shown that the same out-product of YD's also corresponds to the inner Kronecker product of two reps of the same linear group.

Let us emphasize that we are talking about the inner Kronecker product for the linear group. (Outer Kronecker products for linear groups will be mentioned briefly later.) Thus the correspondence between the words describing the Kronecker products and those describing YD operations no longer holds when we go to the linear groups. This is why the names of the YD operations were chosen to be reminiscent of but not identical to the standard terminology for Kronecker products. Nevertheless it is a remarkable and useful fact that the same table of YD out-products has these two interpretations.

Recall that a given linear group has reps labelled by YD's of any number of boxes  $\#[\lambda]$ . The only limitation on the acceptability of YD's which depends on the particular linear group involved is on the number of



non-vanishing rows:  $NR[\lambda] \leq n$  for the groups  $GL_n$ ,  $U_n$ ,  $SU_n$ , etc. Thus our statement about out-products of YD's implies that the inner Kronecker products of reps of  $GL_n$ , say, are given by the same data as those of  $GL_n$ , or  $U_n$ , or  $SU_n$ , and in fact are independent of  $n$ . This is true with one minor qualification which we now explain.

To obtain the inner Kronecker product of two reps of  $GL_n$  (or  $U_n$  or  $SU_n$  or ...) labelled by  $[\lambda]$  and  $[\lambda']$ , one constructs or looks up the out-product  $[\lambda] \circ [\lambda']$ . In general this will contain several YD's  $[\lambda'']$ , some of which are not acceptable as reps of  $GL_n$  because  $NR[\lambda''] > n$ . These unacceptable ones are simply to be dropped and the acceptable ones provide the answer sought. The process of eliminating the unacceptable  $[\lambda'']$  might be called sifting, with mesh  $n$ . It is the only part of the calculation that depends on  $n$  and since it is so trivial no new notation need be introduced: it will henceforth be assumed that out-products have been sifted with appropriate mesh for the group involved.

As an example consider the inner Kronecker product of the reps of  $U_7$  labelled  $[31]$  and  $[21^2]$ . From the calculation above, it is a direct sum of the following reps of  $U_7$ :  $[521] + [51^3] + [431] + 2[421^2] + [42^2] + [41^4] + [3^21^2] + [32^21] + [321^3]$ . But if we want the inner Kronecker product of the reps of  $GL_4$  labelled  $[31]$  and  $[21^2]$ , this must now be sifted with mesh 4, eliminating  $[41^4]$  and  $[321^3]$ . The answer is  $[521] + [51^3] + [431] + 2[421^2] + [42^2] + [3^21^2] + [32^21]$ . The corresponding result for  $U_3$  is  $[521] + [431] + [42^2]$ .

Now this linear-group interpretation of out-products implies a dimensional check,

$$\dim U_n[\lambda] \dim U_n[\lambda'] = \sum_{[\lambda'']} t_{\lambda \lambda' \lambda''} \dim U_n[\lambda''], \quad \underline{2.26}$$





quite different from the one already given. In fact this equation provides an infinite number of dimensional checks, one for each value of  $n$ . We illustrate this with the second example worked out above.

$$[2^2] \circ [21] = [43] + [421] + [3^2 1] + [32^2] + [321^2] + [2^3 1].$$

Sifting leaves this unaltered for  $n \geq 4$ . Thus we get (using Table 2.1) the following sample dimensional checks.

$$n = 4: \quad 20 \times 20 = 400 = 140 + 140 + 60 + 36 + 20 + 4;$$

$$n = 5: \quad 50 \times 40 = 2000 = 560 + 700 + 315 + 210 + 175 + 40; \text{ etc.}$$

For the groups  $U_3$  or  $SU_3$  this becomes

$$[2^2] \circ [21] = [43] + [421] + [3^2 1] + [32^2];$$

$$6 \times 8 = 48 = 24 + 15 + 6 + 3.$$

And for  $U_2$  or  $SU_2$ , sifting with mesh 2 leaves only

$$[2^2] \circ [21] = [43];$$

$$1 \times 2 = 2.$$

Sifting does not interfere with dimensional checks since we set  $\dim U_n[\lambda] = 0$  if  $NR[\lambda] > n$ .

This discussion, like Section 2.1, ends with an example from the well known quantum theory of angular momentum. We have seen that for  $SU_2$  the reps  $[2^2]$ ,  $[21]$ ,  $[43]$  are those usually labelled  $j = \frac{1}{2}(\lambda_1 - \lambda_2) = 0, \frac{1}{2}, \text{ and } \frac{1}{2}$  respectively ( $[21]$  and  $[43]$  being equivalent for  $SU_2$ ). Thus  $[2^2] \circ [21] = [43]$  verifies in a trivial case the well known Clebsch-Gordan series for  $SU_2$  or  $R_3$ :

$$D_{j_1} \dot{\times} D_{j_2} = D_{j_1+j_2} + D_{j_1+j_2-1} + \dots + D_{|j_1-j_2|}, \quad 2.27$$

$$\therefore D_0 \dot{\times} D_{\frac{1}{2}} = D_{\frac{1}{2}}.$$

A less trivial example is

$$\begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \end{array} \circ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & 1 & 1 & \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & 1 & \\ \hline 1 & & & & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \\ \hline 1 & 1 & & & & \\ \hline \end{array},$$



$$[5] \circ [2] = [7] + [61] + [52],$$

$$6 \times 3 = 8 + 6 + 4,$$

$$D_{5/2} \times D_1 = D_{7/2} + D_{5/2} + D_{3/2}.$$

Sifting with mesh 1 instead of 2 as above would have given the erroneous result  $D_{5/2} \times D_1 = D_{7/2}$ . This illustrates that for  $SU_n$  or  $SL_n$  one must sift with mesh  $n$  and not  $n-1$ : even though the rep classes of these special-unimodular groups correspond in some sense to the YD's of not more than  $n-1$  non-vanishing rows, it is simpler to emphasize that their admissible reps are given by the YD's of not more than  $n$  non-vanishing rows, as advocated in Section 2.1.

Finally we must define a more specialized concept related to out-products, and which is needed later, the Foulkes ('differential') operator. If

$$[\lambda] \circ [\lambda'] = \sum_{[\lambda'']} t_{\lambda \lambda' \lambda''} [\lambda'']$$

defines  $t_{\lambda \lambda' \lambda''}$ , the Foulkes operator  $D([\lambda'])$  applied to a YD  $[\lambda'']$  gives

$$D([\lambda']) [\lambda''] = \sum_{[\lambda]} t_{\lambda \lambda' \lambda''} [\lambda]. \quad \underline{2.28}$$

In words,  $[\lambda]$  occurs in the expression  $D([\lambda']) [\lambda'']$  exactly as often as  $[\lambda'']$  occurs in the out-product  $[\lambda] \circ [\lambda']$ . This implies that

$$\#[\lambda] = \#[\lambda''] - \#[\lambda']. \quad \underline{2.29}$$

The operator  $D([\lambda'])$ , applied to a YD  $[\lambda'']$ , produces (in general) several YD's  $[\lambda]$  of fewer boxes than  $\#[\lambda'']$ .

Instead of a single YD  $[\lambda']$  in the parentheses, we can also have a sum or an out-product, in which case

$$D([\mu] + [\mu']) = D([\mu]) + D([\mu']), \quad \underline{2.30}$$

$$D([\mu] \circ [\mu']) = D([\mu]) D([\mu']); \quad \underline{2.31}$$

in 2.31 juxtaposition denotes successive application of the Foulkes operators from right to left. These properties follow from the fact that



the out-product of YD's is distributive with respect to the addition of YD's.

As an example, the only out-products leading to YD's of three boxes are

$$\begin{aligned}[2] \circ [1] &= [3] + [21], \\ [1^2] \circ [1] &= [21] + [1^3].\end{aligned}$$

From this it follows that

$$\begin{aligned}D([1])[3] &= [2], & D([2])[3] &= [1], \\ D([1])[21] &= [2] + [1], & D([2])[21] &= [1], \\ D([1])[1^3] &= [1^2]; & D([2])[1^3] &= 0;\end{aligned}$$

and so on.

The following statements summarize this section. The inner Kronecker product of two reps of the same symmetric group is given by the in-product of YD's. The outer Kronecker product of reps of two symmetric groups is given by the out-product of YD's. The inner Kronecker product of two reps of the same linear group is also given by the (sifted) out-product of YD's. Foulkes operators produce something like the inverse of out-products. Outer Kronecker products of reps of two different linear groups will not be examined in detail, but the duality between symmetric and linear groups, as discussed for instance by Robinson (R61), implies that they are given by the in-product of YD's.

These facts and notations, and a few others from following sections are summarized in Table 2-2, which can be used as an index to YD operations when these are encountered in later chapters.



TABLE 2-2    Summary of YD Operations

<u>Notation</u>	<u>YD Operation</u>	<u>Page</u>	<u>Applications</u>	<u>Number of Boxes</u>
$[\alpha] + [\beta]$	sum	24	direct sum of reps	$\# \{ [\alpha] + [\beta] \} = \# [\alpha] \quad \text{if} \quad \# [\alpha] = \# [\beta]$
$[\alpha] \cdot [\beta]$	in-product	24	inner Kronecker product of reps of a symmetric group; <u>also</u> outer Kronecker product of reps of linear groups	$\# \{ [\alpha] \cdot [\beta] \} = \# [\alpha] = \# [\beta]$
$[\alpha] \circ [\beta]$	out-product	28	outer Kronecker product of reps of symmetric groups; <u>also</u> inner Kronecker product of reps of a linear group	$\# \{ [\alpha] \circ [\beta] \} = \# [\alpha] + \# [\beta]$
$D([\alpha]) \rangle [\beta]$	Foulkes operator	33	(related to the out-product operation)	$\# \{ D([\alpha]) \rangle [\beta] \} = \# [\beta] - \# [\alpha]$
$[\alpha] \overline{\circ} [\beta]$	out-plethysm	38	(ordinary) plethysm for linear groups; <u>also</u> part of out-power	$\# \{ [\alpha] \overline{\circ} [\beta] \} = \# [\alpha] \# [\beta]$
$[\alpha] \div [\beta]$	in-plethysm	42	(inner) plethysm for symmetric groups; <u>also</u> part of in-power	$\# \{ [\alpha] \div [\beta] \} = \# [\alpha]$





### 2.3 Plethysm, and the Out-Plethysm of YD's

The concept of subduction defined in Section 2.1 will be of central importance for applications. In this section are reviewed some terminology, notation and techniques used in evaluating subductions. Given a rep of a linear group, the problem is to find into what reps it breaks down when we confine our attention to a subgroup, i.e. to obtain the reduction of the subduced representation as a direct sum of reps of the subgroup.

Let  $H$  be a finite or compact group, and  $\Gamma_f^H$  an  $f$ -dimensional representation of  $H$ . The  $f \times f$  matrices which constitute the representation  $\Gamma_f^H$  can themselves be considered as elements of a group: in fact they form a subgroup of  $GL_f$ , by construction a homomorphic image of  $H$ , and with matrix multiplication as its group operation. Now consider a rep  $[\lambda]$  of the full linear group  $GL_f$ : it maps every element of  $GL_f$ , i.e. every non-singular  $f \times f$  matrix, onto some representative matrix of dimension  $\dim U_f[\lambda]$ . In particular it maps those  $f \times f$  matrices of  $\Gamma_f^H$  onto representative matrices which form a representation, the subduced representation, (in general reducible), of the subgroup  $\Gamma_f^H \subset GL_f$ . We denote this subduced representation, of dimension  $\dim U_f[\lambda]$ , by  $\Gamma_f^H \bar{p} [\lambda]$ . Since the group  $\Gamma_f^H$  is a homomorphic image of the group  $H$ ,  $\Gamma_f^H \bar{p} [\lambda]$  is also a representation of  $H$ , and can be expressed as a direct sum of reps of  $H$ .  $\Gamma_f^H \bar{p} [\lambda]$  is read ' $\Gamma_f^H$  plethys  $[\lambda]$ ' and is called a plethysm. If  $H$  is one of the groups mentioned in Section 2.1, techniques for actually obtaining the reduction of  $\Gamma_f^H \bar{p} [\lambda]$  as a direct sum of reps of  $H$  exist in principle, although they are unsatisfactory for complicated cases.

Before giving examples let us summarize all this from a different point of view. Effectively we have taken a rep of a group  $GL_f$  and we have



performed the subduction  $GL_f \rightarrow H$ , or at least  $GL_f \rightarrow \Gamma_f^H$ , where  $\Gamma_f^H$  is a homomorphic image of  $H$ . Roughly speaking we have asked how the rep  $[\lambda]$  of  $GL_f$  breaks down when it is considered as a representation of  $H$ . Or in terms of the basis functions which form the carrier space of  $[\lambda]$ , (the linear vector space on which the matrices of the rep  $[\lambda]$  can be imagined to operate), we have broken down this linear vector space into subspaces invariant under the operations of  $H$  (though it contained no subspaces invariant under the operations of the full linear group  $GL_f$ ); this amounts to saying that we have obtained a subclassification of the basis functions of  $[\lambda]$ . To do this we had to refer to a representation  $\Gamma_f^H$  of  $H$ , chosen to have dimension  $f$ . If we approach the problem as described in this paragraph rather than the previous one, i.e. from a given  $f$ ,  $[\lambda]$ , and  $H$ , the choice of  $\Gamma_f^H$  is made by considering the basis functions of the rep  $[1]$  of  $GL_f$ . Remembering that  $\dim U_f[1] = f$ , we use these as basis functions for  $\Gamma_f^H$  also, and obtain  $\Gamma_f^H$  in principle by asking in what way these basis functions transform under the operations of the group  $H$ ; (in practice we need only identify  $\Gamma_f^H$ , not write down its matrices). The answer involves the physical interpretation of  $f$ ,  $[\lambda]$ , and  $H$ , as examples will show in the following chapters.

But let us return now to the formal problem of evaluating plethysms. We consider first the case where  $H$  is a linear group ( $GL_n$ ,  $U_n$ ,  $SU_n$ , etc.), and in the next section the case  $H = S_k$ . If  $H$  is a linear group its representation  $\Gamma_f^H$  is labelled by a YD, say  $[\mu]$ , and we now write it  $\Gamma_{[\mu]}^H$ , with  $\dim U_n[\mu] = f$ . (The more general case where  $\Gamma_f^H$  is reducible, corresponding to a linear combination of YD's instead of a single  $[\mu]$ , will not be discussed here.) We consider therefore

$$\Gamma_{[\mu]}^H \bar{p} [\lambda] = \sum_{[\nu]} k_{\mu\lambda\nu} \Gamma_{[\nu]}^H. \quad \underline{2.32}$$



Once again it is convenient to define an operation on YD's by

$$[\mu] \overline{\circ} [\lambda] = \sum_{[\nu]} k_{\mu\lambda\nu} [\nu], \quad \underline{2.33}$$

with the same coefficients  $k_{\mu\lambda\nu}$ ; this will be called the out-plethysm of YD's. The reason is that, much like the out-product to which it is closely related as we see below, the out-plethysm is independent of the particular choice of  $H$  ( $GL_n$ ,  $SL_n$ ,  $U_n$ ,  $SU_n$ , etc.), and depends on  $n$  only in that it must be sifted with mesh  $n$ . Since the YD's  $[\nu]$  designate reps of  $H$ , only those with  $NR[\nu] \leq n$  are acceptable when the out-plethysm is to be used for the subduction  $GL_f \rightarrow SU_n$ , say.

It can be shown that only YD's  $[\nu]$  with  $\#[\nu] = \#[\mu] \#[\lambda]$  occur in the direct sums. There is no general formula, and the evaluation of non-trivial out-plethysms requires artful blending of various methods so as to keep the calculations from getting out of hand. Explicit tables are given by Ibrahim (Ib50) for  $\#[\nu] \leq 18$  on the right hand side. Only a few special cases have been calculated beyond this range. Since these tables are not readily available a few special cases gleaned from various sources are reproduced in Table 2-3.

The following dimensional check follows from the explanations given:

$$\dim U_f[\lambda] = \sum_{[\nu]} k_{\mu\lambda\nu} \dim U_n[\nu], \quad \underline{2.34}$$

with  $f = \dim U_n[\mu]$ . As in the case of out-products this provides for each out-plethysm of YD's any number of different dimensional checks since it applies for any value of  $n$ .

As an example consider

$$[2] \overline{\circ} [21] = [51] + [42] + [321].$$

Suppose this is being used in a problem where  $H$  is  $SU_3$ , i.e.  $n=3$ . Then





TABLE 2-3 Examples of Out-Plethysm

$[0]\overline{\circ}[\lambda] = [\mu]\overline{\circ}[0] = [0].$	$[1]\overline{\circ}[\lambda] = [\lambda].$	$[\mu]\overline{\circ}[1] = [\mu].$	$[2]\overline{\circ}[2] = [4] + [2^2].$
$[2]\overline{\circ}[3] = [6] + [42] + [2^3].$	$[2]\overline{\circ}[4] = [8] + [62] + [4^2] + [42^2] + [2^4].$		$[2]\overline{\circ}[1^2] = [31].$
$[2]\overline{\circ}[21] = [51] + [42] + [321].$	$[2]\overline{\circ}[31] = [71] + [62] + [53] + [521] + [431] + [42^2] + [32^21].$		
$[2]\overline{\circ}[1^3] = [41^2] + [3^2].$	$[2]\overline{\circ}[2^2] = [62] + [521] + [4^2] + [42^2] + [3^21^2].$		
$[2]\overline{\circ}[1^4] = [51^3] + [431].$	$[2]\overline{\circ}[21^2] = [61^2] + [521] + [53] + [431] + [421^2] + [3^22].$		
$[2]\overline{\circ}[5] = [10] + [82] + [64] + [62^2] + [4^22] + [42^3] + [2^5].$	$[2]\overline{\circ}[1^5] = [61^4] + [531^2] + [4^22].$		
$[2]\overline{\circ}[41] = [91] + [82] + [73] + [721] + [64] + [631] + [62^2] + [541] + [532] + [52^21] + [4^22] + [4321] + [42^3] + [32^31].$			
$[2]\overline{\circ}[32] = [82] + [73] + [721] + [64] + [631] + 2[62^2] + [541] + [532] + [531^2] + [52^21] + [4^22] + [4321] + [42^3] + [32^31].$			
$[2]\overline{\circ}[31^2] = [81^2] + [73] + [721] + 2[631] + [621^2] + [5^2] + [541] + 2[532] + [531^2] + [52^21] + [4^21^2] + [43^2] + [4321] + [42^21^2] + [3^22^2].$			
$[2]\overline{\circ}[2^21] = [721] + [64] + [631] + [62^2] + [621^2] + [541] + [532] + [531^2] + [52^21] + [4^22] + [4321] + [431^3] + [3^31].$			
$[2]\overline{\circ}[21^3] = [71^3] + [621^2] + [631] + [541] + [532] + [531^2] + [521^3] + [521^2] + [4^21^2] + [43^2] + [4321].$			
$[2]\overline{\circ}[6] = [12] + [102] + [84] + [82^2] + [6^2] + [642] + [62^3] + [4^3] + [4^22^2] + [42^4] + [2^6].$			
$[3]\overline{\circ}[2] = [6] + [42].$	$[3]\overline{\circ}[3] = [9] + [72] + [63] + [52^2] + [4^21].$	$[3]\overline{\circ}[1^3] = [71^2] + [63] + [531] + [3^3].$	
$[3]\overline{\circ}[1^2] = [51] + [3^2].$	$[3]\overline{\circ}[21] = [81] + [72] + [63] + [621] + [54] + [531] + [432].$		
$[3]\overline{\circ}[4] = [12] + [102] + [93] + [84] + [82^2] + [741] + [732] + [6^2] + [642] + [62^3] + [5421] + [4^3].$			
$[4]\overline{\circ}[2] = [8] + [62] + [4^2].$	$[4]\overline{\circ}[1^2] = [71] + [53].$		
$[4]\overline{\circ}[3] = [12] + [102] + [93] + [84] + [82^2] + [741] + [642] + [4^3] + [6^2].$			



$f = \dim U_3[2] = \frac{1}{2} \times 3 \times 4 = 6$ , and  $[21]$  designates a rep of  $GL_6$ ; we want the reduction of its subduced representation into a direct sum of reps of  $SU_3$ . The dimension of the subduced representation is  $\dim U_6[21] = 70$  (from Table 2-1) and we have on the right hand side  $\dim U_3[51] + \dim U_3[42] + \dim U_3[321] = 35 + 27 + 8 = 70$ .

If the same out-plethysm is used with  $H = U_2$ , it must be sifted with mesh 2, leaving only

$$[2]\overline{o}[21] = [51] + [42].$$

Now  $f = \dim U_2[2] = 3$ , and the subduction involved is  $GL_3 \rightarrow U_2$ . We have  $\dim U_3[21] = 8$ , and  $\dim U_2[51] + \dim U_2[42] = 5 + 3$ .

Because of the duality which permeates this subject, it is not surprising that the out-plethysm of YD's has another interpretation, in terms of reps of symmetric groups. Consider the out-product of  $[\mu]$  by itself  $a$  times,

$$[\mu] \circ [\mu] \circ \dots \circ [\mu],$$

which one might call the  $a$ 'th out-power of  $[\mu]$ . Since  $[\mu]$  designates a rep of  $S_k$ , where  $k = \#[\mu]$ , this  $a$ 'th out-power is a representation, in general reducible, of  $S_{ak}$ . If we express it as a direct sum of reps of  $S_{ak}$  we obtain a linear combination of YD's  $[\nu]$  with  $\#[\nu] = ak = a\#[\mu]$ . But we can also classify these YD's  $[\nu]$ , or rather the basis functions of the reps of  $S_{ak}$  which they label, according to their behaviour under operations of the group  $S_a$ ; they will belong to one or another of the symmetry types  $[\lambda]$  with  $\#[\lambda] = a$ . It turns out that those YD's  $[\nu]$  of the  $a$ 'th out-power of  $[\mu]$  that belong to the symmetry type  $[\lambda]$  are given precisely by the out-plethysm  $[\mu]\overline{o}[\lambda]$ , taken  $\dim S_a[\lambda]$  times. This implies that

$$[\mu] \circ [\mu] \circ \dots \circ [\mu] = \sum_{[\lambda]} [\mu]\overline{o}[\lambda] \dim S_a[\lambda]. \quad \underline{2.35}$$

We have for example the out-product (or second out-power of  $[3]$ ):



$$\begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \circ \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline & & & 1 & 1 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|} \hline & & & 1 & 1 \\ \hline 1 & & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline 1 & 1 & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & & \\ \hline 1 & 1 & 1 & \\ \hline \end{array},$$

$$[3] \circ [3] = [6] + [51] + [2] + [3^3].$$

The so-called symmetrized square (of type  $[2]$  under  $S_2$ ) and anti-symmetrized square (of type  $[1^2]$  under  $S_2$ ) are given by the out-plethysms  $[3] \overline{\circ} [2] = [6] + [42]$ , and  $[3] \overline{\circ} [1^2] = [51] + [3^2]$  respectively.

This interpretation provides yet another dimensional check applicable to out-plethysms (R49). If

$$[\mu] \overline{\circ} [\lambda] = \sum_{[\nu]} k_{\mu\lambda\nu} [\nu],$$

$$\#[\mu] = k,$$

$$\text{and } \#[\lambda] = a,$$

$$\text{then } \frac{(ak)!}{(k!)^a a!} \{ \dim S_k[\mu] \}^a \dim S_a[\lambda] = \sum_{[\nu]} k_{\mu\lambda\nu} \dim S_{ak}[\nu]. \quad \underline{2.36}$$

This must of course be applied to the unsifted out-plethysm. For

$[2] \overline{\circ} [21] = [51] + [42] + [321]$ , the example to which other checks have already been applied, this gives

$$\frac{6!}{(2!)^3 3!} (1)^3 2 = 30 = 5 + 9 + 16.$$

This completes the discussion of plethysm in the case where  $H$  is a linear group; in the next section we take up the case where  $H$  is a symmetric group.

## 2.4 The In-Plethysm of YD's

The preceding section started with a general description of plethysm and then considered the special case where  $H$  is a linear group; this case is sometimes described as ordinary plethysm. An operation on YD's called out-plethysm was defined in such a way that the sifted out-plethysm





$[\mu]\overline{o}[\lambda]$  provides the corresponding plethysm  $\Gamma_{[\mu]}^H \overline{p}[\lambda]$  for any linear group  $H$ .

In this section the general description of plethysm is still relevant, and we consider the second (and last) special case:  $H$  is now taken to be the symmetric group  $S_k$ . This means that we are aiming at a subduction  $GL_f \rightarrow \Gamma_{[\mu]}^H$ , where the subgroup  $\Gamma_{[\mu]}^H \subset GL_f$  is a homomorphic image of  $S_k$ ; loosely speaking, we investigate the subduction  $GL_f \rightarrow S_k$ . The subgroup  $\Gamma_{[\mu]}^H$  is of course an  $f$ -dimensional representation of  $S_k$ . Forsaking the general case where  $\Gamma_{[\mu]}^H$  is reducible, we take  $\Gamma_{[\mu]}^H$  to be a rep of  $S_k$ , hence labelled by a single YD  $[\mu]$  of  $k$  boxes, with  $\dim S_k[\mu] = f$ . This situation is usually described as inner plethysm, and we use it to define an operation on YD's called the in-plethysm of YD's. If

$$\Gamma_{[\mu]}^{S_k} \overline{p}[\lambda] = \sum_{[\nu]} G_{\mu\lambda\nu} \Gamma_{[\nu]}^{S_k}, \quad 2.37$$

the in-plethysm of YD's is defined by

$$[\mu] \tau [\lambda] = \sum_{[\nu]} G_{\mu\lambda\nu} [\nu], \quad 2.38$$

with the same coefficients  $G_{\mu\lambda\nu}$ . Here  $\#[\nu] = \#[\mu] = k$ , and the dimensional check is

$$\dim U_f[\lambda] = \sum_{[\nu]} G_{\mu\lambda\nu} \dim S_k[\nu], \quad 2.39$$

with  $f = \dim S_k[\mu]$ .

An example is  $[31] \tau [21] = [2^2] + [21^2] + [31]$ . Here  $f = \dim S_4[31] = 3$  so that  $[21]$  labels a rep of  $GL_3$ . The reduction of its subduced representation into reps of  $S_4$  is checked as follows:

$$\begin{array}{ccccccc} \dim U_3[21] & = & \dim S_4[2^2] & + & \dim S_4[21^2] & + & \dim S_4[31], \\ 8 & = & 2 & + & 3 & + & 3. \end{array}$$

There is no practical procedure for evaluating general non-trivial





in-plethysms. There is a definite procedure however for in-plethysms of the form  $[c1] \dot{-} [\lambda]$ , such as the ones needed for the physical application discussed in this thesis. (The notation  $[c\beta] = [\mu]$  introduced in Section 2.2 leaves  $\mu_1$  as an undetermined variable to be fixed by the context: e.g.  $[c1] = [71]$  if the context prescribes  $\#[c1] = 8$ .) The procedure is given by the following rather formidable theorem due to Littlewood (L58).

The theorem describes the construction of a general operator  $\mathcal{D}$ , (which is not itself a Foulkes operator though it includes Foulkes operators  $D([\alpha])$ ), and claims that if

$$\mathcal{D}[\rho] = \sum_{[\sigma]} V_{\rho\sigma} [\sigma], \quad \underline{2.40}$$

then with the same coefficients  $V_{\rho\sigma}$ ,

$$[c1] \dot{-} [\rho] = \sum_{[\sigma]} V_{\rho\sigma} [c\sigma]. \quad \underline{2.41}$$

As in the case of the semi-general formulas for in-products, some of the terms  $[c\sigma]$  so obtained will be generalized YD's, which must be handled according to the modification rules of Section 2.2. The  $[\sigma]$  themselves however, produced by the application of  $\mathcal{D}$  to  $[\rho]$ , must be YD's satisfying  $\sigma_1 \geq \sigma_2 \geq \dots \geq 0$ , not generalized YD's.

The operator  $\mathcal{D}$  is as follows:

$$\mathcal{D} = \sum_{\ell_2=0}^{\infty} \sum_{[\lambda^{\ell_2}]} \sum_{\ell_3=0}^{\infty} \sum_{[\lambda^{\ell_3}]} \dots \sum_{r_2=0}^{\infty} \sum_{r_3=0}^{\infty} \dots \mathcal{D}_i, \quad \underline{2.42}$$

$$\text{with } \mathcal{D}_i = [\lambda^{\ell_2}] \circ [\lambda^{\ell_3}] \circ \dots \circ D([2] \bar{\circ} [\lambda^{\ell_2}]) D([3] \bar{\circ} [\lambda^{\ell_3}]) \dots D([2] \bar{\circ} [r_2]) D([3] \bar{\circ} [r_3]) \dots \quad \underline{2.43}$$

Here  $\bar{\circ}$  denotes out-plethysm and  $\circ$  out-product;  $D([\alpha])$  denotes a Foulkes operator as in Section 2.2, but the  $[\alpha]$  of the Foulkes operators used here are out-plethysms to be evaluated (hence in general linear combinations of YD's) rather than explicitly given single YD's. The  $[\lambda^{\ell_1}]$  are



partitions of the corresponding integers  $\ell_i$  (i.e.  $\#[\lambda^{\ell_i}] = \ell_i$ ), and each  $\sum_{[\lambda^{\ell_i}]}$  runs over all the partitions of  $\ell_i$ . Each term  $\mathcal{D}_i$  in this multiple sum corresponds to a choice of one infinite set of non-negative integers  $r_2, r_3, \dots, r_j, \dots$ , of another infinite set of non-negative integers  $\ell_2, \ell_3, \dots, \ell_j, \dots$ , and of a partition  $[\lambda^{\ell_i}]$  of each  $\ell_i$ . Thus  $\mathcal{D}$  has a multiple infinity of terms and we must learn to list them in such an order that only the first few will give non-vanishing contributions when  $\mathcal{D}$  is applied to a YD  $[\rho]$ .

For this purpose, Kretzschmar (K60b) has defined the expansion

$$\mathcal{D} = \sum_{\ell=0}^{\infty} \Delta_{\ell}, \quad \underline{2.45}$$

by the requirement that  $\Delta_{\ell}[\rho] = \sum_{[\sigma]} V_{\rho\sigma}[\sigma]$  should produce only YD's  $[\sigma]$  of  $\#[\sigma] = \#[\rho] - \ell$  boxes, and has given  $\Delta_0$ ,  $\Delta_1$ , and  $\Delta_2$  explicitly. (This summation index  $\ell$  is not one of the  $\ell_i$  of the general  $\mathcal{D}$  formula.)

To see how  $\Delta_{\ell}$  is obtained, recall the effect of various operations on the number of boxes  $\#[\lambda]$ , as summarized in Table 2-2. In this table, the symbol  $\#$  is generalized slightly as follows: if  $\Lambda$  is a linear combination of YD's, then  $\#\{\Lambda\}$  denotes the number of boxes in each of the YD's of  $\Lambda$ ; this implies that they all have the same number, of course.

Consider  $\mathcal{D}_i$ , one of the terms of the multiple sum which constitutes  $\mathcal{D}$ . As we examine  $\mathcal{D}_i$  from right to left, we note that the number of boxes of the (linear combination of) YD's obtained, starting from  $\#[\rho]$ , is first decreased by the action of Foulkes operators, then increased again as we form out-products. In fact, using the last column of Table 2-2,

$$\begin{aligned} \#\{\mathcal{D}_i[\rho]\} &= \#[\rho] - \sum_{j=2}^{\infty} \#\{[j]\overline{o}[r_j]\} - \sum_{j=2}^{\infty} \#\{[j]\overline{o}[\lambda^{\ell_j}]\} + \sum_{j=2}^{\infty} \#[\lambda^{\ell_j}] = \\ &= \#[\rho] - \sum_{j=2}^{\infty} (jr_j + j\ell_j - \ell_j). \end{aligned} \quad \underline{2.46}$$









	$\ell_2$	$r_2$	$\ell_3$	$r_3$	$\ell_4$	$r_4$	$\ell_5$	$r_5$	$\ell_6$	$\leftarrow \ell_j \text{ or } r_j$ $\leftarrow jr_j \text{ or } (j-1)\ell_j$	$\Delta_\rho$	<u>2.47</u>
	1	2	2	3	3	4	4	5	5			

5	0	0	0	0	0	0	0	1	0	$D([5]\bar{o}[1]) + [1] \circ D([6]\bar{o}[1]) + [1] \circ D([2]\bar{o}[1])D([4]\bar{o}[1])$ $+ [1] \circ D([2]\bar{o}[1])[1] \circ D([5]\bar{o}[1]) + D([2]\bar{o}[1])D([3]\bar{o}[1])$ $+ [1] \circ D([3]\bar{o}[1])D([3]\bar{o}[1]) + [1] \circ D([4]\bar{o}[1])D([2]\bar{o}[1])$ $+ [1] \circ D([3]\bar{o}[1])[1] \circ D([4]\bar{o}[1]) + [2] \circ D([2]\bar{o}[2])D([3]\bar{o}[1])$ $+ [1^2] \circ D([2]\bar{o}[1^2])D([3]\bar{o}[1]) + [2] \circ D([2]\bar{o}[2])[1] \circ D([4]\bar{o}[1])$ $+ [1^2] \circ D([2]\bar{o}[1^2])[1]D([4]\bar{o}[1]) + [1] \circ D([2]\bar{o}[1])D([2]\bar{o}[2])$ $+ [1] \circ D([2]\bar{o}[1])[1] \circ D([3]\bar{o}[1])D([2]\bar{o}[1])$ $+ [1] \circ D([2]\bar{o}[1])[2] \circ D([3]\bar{o}[2])$ $+ [1] \circ D([2]\bar{o}[1])[1^2] \circ D([3]\bar{o}[1^2]) + [3] \circ D([2]\bar{o}[3])D([2]\bar{o}[1])$ $+ [21] \circ D([2]\bar{o}[21])D([2]\bar{o}[1]) + [1^3] \circ D([2]\bar{o}[1^3])D([2]\bar{o}[1])$ $+ [3] \circ D([2]\bar{o}[3])[1] \circ D([3]\bar{o}[1])$ $+ [21] \circ D([2]\bar{o}[21])[1] \circ D([3]\bar{o}[1])$ $+ [1^3] \circ D([2]\bar{o}[1^3])[1] \circ D([3]\bar{o}[1]) + [5] \circ D([2]\bar{o}[5])$ $+ [41] \circ D([2]\bar{o}[41]) + [32] \circ D([2]\bar{o}[32]) + [31^2] \circ D([2]\bar{o}[31^2])$ $+ [2^21] \circ D([2]\bar{o}[2^21]) + [21^3] \circ D([2]\bar{o}[21^3]) + [1^5] \circ D([2]\bar{o}[1^5]).$		
	0	0	0	0	0	0	0	0	1			
	1	0	0	0	0	1	0	0	0			
	1	0	0	0	0	0	1	0	0			
	0	1	0	1	0	0	0	0	0			
	0	0	1	1	0	0	0	0	0			
	0	1	0	0	1	0	0	0	0			
	0	0	1	0	1	0	0	0	0			
	2	0	0	1	0	0	0	0	0			
	2	0	0	0	1	0	0	0	0			
	1	2	0	0	0	0	0	0	0			
	1	1	1	0	0	0	0	0	0			
	1	0	2	0	0	0	0	0	0			
	3	1	0	0	0	0	0	0	0			
	3	0	1	0	0	0	0	0	0			
	5	0	0	0	0	0	0	0	0			

The final expressions for the  $\Delta_\rho$  are obtained from these by using Table 2-3 and 2.30:

$$\Delta_0 = 1.$$

$$\Delta_1 = [1] \circ D([2]).$$

$$\Delta_2 = D([2]) + [1] \circ D([3]) + [2] \circ D([4]) + [2] \circ D([2^2]) + [1^2] \circ D([31]).$$

$$\begin{aligned} \Delta_3 = & D([3]) + [1] \circ D([4]) + [1] \circ D([2])D([2]) + [1] \circ D([2])[1] \circ D([3]) + [3] \circ D([6]) \\ & + [3] \circ D([42]) + [3] \circ D([2^3]) + [21] \circ D([51]) + [21] \circ D([42]) + [21] \circ D([321]) \\ & + [1^3] \circ D([41^2]) + [1^3] \circ D([3^2]). \end{aligned}$$

2.48



$$\begin{aligned}
\Delta_4 = & D([4]) + [1] \circ D([5]) + [1] \circ D([2])D([3]) + [1] \circ D([2])[1] \circ D([4]) + D([4]) + D([2^2]) \\
& + [1] \circ D([3])D([2]) + [2] \circ D([6]) + [2] \circ D([42]) + [1^2] \circ D([51]) + [1^2] \circ D([3^2]) \\
& + [2] \circ D([4])D([2]) + [2] \circ D([2^2])D([2]) + [1^2] \circ D([31])D([2]) + [2] \circ D([4])[1] \circ D([3]) \\
& + [2] \circ D([2^2])[1] \circ D([3]) + [1^2] \circ D([31])[1] \circ D([3]) + [4] \circ D([8]) + [4] \circ D([62]) \\
& + [4] \circ D([4^2]) + [4] \circ D([42^2]) + [4] \circ D([2^4]) + [31] \circ D([71]) + [31] \circ D([62]) \\
& + [31] \circ D([53]) + [31] \circ D([521]) + [31] \circ D([431]) + [31] \circ D([42^2]) + [31] \circ D([32^21]) \\
& + [2^2] \circ D([62]) + [2^2] \circ D([521]) + [2^2] \circ D([4^2]) + [2^2] \circ D([42^2]) + [2^2] \circ D([3^21^2]) \\
& + [21^2] \circ D([61^2]) + [21^2] \circ D([521]) + [21^2] \circ D([53]) + [21^2] \circ D([431]) + [21^2] \circ D([421^2]) \\
& + [21^2] \circ D([3^22]) + [1^4] \circ D([51^3]) + [1^4] \circ D([431]).
\end{aligned}$$

$$\begin{aligned}
\Delta_5 = & D([5]) + [1] \circ D([6]) + [1] \circ D([2])D([4]) + [1] \circ D([2])[1] \circ D([5]) + D([2])D([3]) \\
& + [1] \circ D([3])D([3]) + [1] \circ D([4])D([2]) + [1] \circ D([3])[1] \circ D([4]) + [2] \circ D([4])D([3]) \\
& + [2] \circ D([2^2])D([3]) + [1^2] \circ D([31])D([3]) + [2] \circ D([4])[1] \circ D([4]) + \\
& + [2] \circ D([2^2])[1] \circ D([4]) + [1^2] \circ D([31])[1] \circ D([4]) + [1] \circ D([2])D([4]) \\
& + [1] \circ D([2])D([2^2]) + [1] \circ D([2])[1] \circ D([3])D([2]) + [1] \circ D([2])[2] \circ D([6]) \\
& + [1] \circ D([2])[2] \circ D([42]) + [1] \circ D([2])[1^2] \circ D([51]) + [1] \circ D([2])[1^2] \circ D([3^2]) \\
& + [3] \circ D([6])D([2]) + [3] \circ D([42])D([2]) + [3] \circ D([2^3])D([2]) + [21] \circ D([51])D([2]) \\
& + [21] \circ D([42])D([2]) + [21] \circ D([321])D([2]) + [1^3] \circ D([41^2])D([2]) + [1^3] \circ D([3^2])D([2]) \\
& + [3] \circ D([6])[1] \circ D([3]) + [3] \circ D([42])[1] \circ D([3]) + [3] \circ D([2^3])[1] \circ D([3]) \\
& + [21] \circ D([51])[1] \circ D([3]) + [21] \circ D([42])[1] \circ D([3]) + [21] \circ D([321])[1] \circ D([3]) \\
& + [1^3] \circ D([41^2])[1] \circ D([3]) + [1^3] \circ D([3^2])[1] \circ D([3]) + [5] \circ D([10]) + [5] \circ D([82]) \\
& + [5] \circ D([64]) + [5] \circ D([62^2]) + [5] \circ D([4^22]) + [5] \circ D([42^3]) + [5] \circ D([2^5]) + [41] \circ D([91]) \\
& + [41] \circ D([82]) + [41] \circ D([73]) + [41] \circ D([721]) + [41] \circ D([64]) + [41] \circ D([631]) \\
& + [41] \circ D([62^2]) + [41] \circ D([541]) + [41] \circ D([532]) + [41] \circ D([5^21]) + [41] \circ D([4^22]) \\
& + [41] \circ D([4321]) + [41] \circ D([42^3]) + [41] \circ D([32^31]) + [32] \circ D([82]) + [32] \circ D([73]) \\
& + [32] \circ D([721]) + [32] \circ D([64]) + [32] \circ D([631]) + 2[32] \circ D([62^2]) + [32] \circ D([541]) \\
& + [32] \circ D([532]) + [32] \circ D([531^2]) + [32] \circ D([52^21]) + [32] \circ D([4^22]) \\
& + [32] \circ D([4321]) + [32] \circ D([42^3]) + [32] \circ D([3^221^2]) + [31^2] \circ D([81^2]) +
\end{aligned}$$



$$\begin{aligned}
& + [31^2] \circ D([73]) + [31^2] \circ D([721]) + 2[31^2] \circ D([631]) + [31^2] \circ D([621^2]) + [31^2] \circ D([5^2]) \\
& + [31^2] \circ D([541]) + 2[31^2] \circ D([532]) + [31^2] \circ D([531^2]) + [31^2] \circ D([52^21]) \\
& + [31^2] \circ D([4^21^2]) + [31^2] \circ D([43^2]) + [31^2] \circ D([4321]) + [31^2] \circ D([42^21^2]) \\
& + [31^2] \circ D([3^22^2]) + [2^21] \circ D([721]) + [2^21] \circ D([64]) + [2^21] \circ D([631]) + [2^21] \circ D([62^2]) \\
& + [2^21] \circ D([621^2]) + [2^21] \circ D([541]) + [2^21] \circ D([532]) + [2^21] \circ D([531^2]) \\
& + [2^21] \circ D([52^21]) + [2^21] \circ D([4^22]) + [2^21] \circ D([4321]) + [2^21] \circ D([431^3]) \\
& + [2^21] \circ D([3^31]) + [21^3] \circ D([71^3]) + [21^3] \circ D([621^2]) + [21^3] \circ D([631]) + [21^3] \circ D([541]) \\
& + [21^3] \circ D([532]) + [21^3] \circ D([531^2]) + [21^3] \circ D([521^3]) + [21^3] \circ D([4^21^2]) \\
& + [21^3] \circ D([43^2]) + [21^3] \circ D([4321]) + [1^5] \circ D([61^4]) + [1^5] \circ D([531^2]) + [1^5] \circ D([4^22]).
\end{aligned}$$

Such calculations are of course done once and for all, and the results are used in the evaluation of any in-plethysm of the form  $[c1] \cdot [\rho]$ . From the definition of the  $\Delta_\ell$  we now see that in applying  $\mathcal{D}$  to  $[\rho]$ , (so as to evaluate  $[c1] \cdot [\rho]$ ), only the first  $\#[\rho]$  terms in the expansion  $\mathcal{D} = \sum_{\ell=0}^{\infty} \Delta_\ell$  will produce YD's  $[\sigma]$  with  $\#[\sigma] \geq 0$ . (By definition the term  $\Delta_\ell$  leads to  $\#[\sigma] = \#[\rho] - \ell$ ). Note that  $[\sigma] = [0]$  must not be omitted: it leads to a non-vanishing term  $[c0]$  in the in-plethysm.

For  $[\rho] = [0]$  we therefore obtain

$$\Delta_0[0] = [0], \quad \Delta_\ell[0] = 0 \text{ for } \ell > 0;$$

$$\text{therefore} \quad [c1] \cdot [0] = [c0].$$

Similarly for  $[\rho] = [1]$  we get

$$\Delta_0[1] = [1], \quad \Delta_1[1] = 0, \quad \Delta_\ell[1] = 0 \text{ for } \ell > 1;$$

$$\text{hence} \quad [c1] \cdot [1] = [c1].$$

For  $[\rho] = [2]$ ,

$$\Delta_0[2] = [2], \quad \Delta_1[2] = [1], \quad \Delta_2[2] = [0];$$

$$\text{therefore} \quad [c1] \cdot [2] = [c2] + [c1] + [c0].$$

Such semi-general formulas have been obtained by Murnaghan (Mu55) by a different method (and with three misprints) for  $\#[\rho] \leq 6$ . The corrected





formulas are reproduced in Table 2-4.  $[\rho]$  is indicated in the column on the left and the coefficients  $V_{\rho\sigma}$  of  $[\sigma]$  in the formula  $[c1] \cdot [\rho] = \sum_{[\sigma]} V_{\rho\sigma} [\sigma]$  appear in the same horizontal row underneath the appropriate column headings  $[\sigma]$ . Thus

$$[c1] \cdot [3] = [c0] + 2[c1] + [c2] + [c1^2] + [c3]. \quad \underline{2.49}$$

The entries inside each of the rectangles indicated are obtained from the same  $\Delta_{\ell}$ , with the  $\ell=0$  rectangle occupying the diagonal position in the table and the other rectangles adjoining it with  $\ell$  increasing from right to left: e.g. the five terms of  $[c1] \cdot [3]$  above come from  $\ell = 3, 2, 1, 1, 0$  respectively.

The final step is to obtain explicit in-plethysms from the semi-general formulas of Table 2-4. This is a straightforward application of the modification rules described in Section 2.2. From  $[c1] \cdot [3]$  for example (2.49), we readily obtain any of the following explicit in-plethysms.

$$\begin{aligned} \text{If } H \text{ is } S_3: \quad [21] \cdot [3] &= [3] + 2[21] + [12] + [1^3] + [03] \\ &= [3] + 2[21] + [1^3] - [21] = [3] + [21] + [1^3], \end{aligned}$$

$$(f = \dim S_3[21] = 2, \text{ and } \dim U_2[3] = 4 = 1+2+1);$$

$$\begin{aligned} \text{if } H \text{ is } S_2: \quad [1^2] \cdot [3] &= [2] + 2[1^2] + [02] + [01^2] + [-1\ 3] \\ &= [2] + 2[1^2] - [1^2] - [21] = [1^2], \end{aligned}$$

$$(f = \dim S_2[1^2] = 1, \text{ and } \dim U_1[3] = 1 = \dim S_2[1^2]);$$

$$\begin{aligned} \text{if } H \text{ is } S_4: \quad [31] \cdot [3] &= [4] + 2[31] + [2^2] + [21^2] + [13] \\ &= [4] + 2[31] + [2^2] + [21^2] - [2^2] \\ &= [4] + 2[31] + [21^2], \end{aligned}$$

$$(f = \dim S_4[31] = 3, \text{ and } \dim U_3[3] = 10 = 1 + 2 \times 3 + 3); \text{ etc.}$$

$$\text{Similarly, from } [c1] \cdot [21] = [c1] + [c2] + [c1^2] + [c21],$$

we obtain, if  $H$  is  $S_4$ ,

$$[31] \cdot [21] = [31] + [2^2] + [21^2] + [121] = [31] + [2^2] + [21^2],$$





TABLE 2-4 In-Plethysms of the Form  $[cl]_r[p]$ 

$[p]$ ↓	$c_0$	$c_1$	$c_2$ $c_1^2$	$c_3$ $c_2^1$ $c_1^3$	$c_4$ $c_3^1$ $c_2^2$ $c_2^1 c_1^2$ $c_1^4$	$c_5$ $c_4^1$ $c_3^2$ $c_3^1 c_1^2$ $c_2^2 c_1$ $c_2^1 c_1^3$ $c_1^5$	$c_6$ $c_5^1$ $c_4^2$ $c_4^1 c_1^2$ $c_3^3$ $c_3^2 c_1$ $c_3^1 c_1^3$ $c_2^3 c_1^2$ $c_2^2 c_1^4$ $c_1^6$
0	1						
1		1					
2 $1^2$	1	1	1 1				
3 $2^1$ $1^3$	1	2 1	1 1 1 1	1 1 1			
4 $3^1$ $2^2$ $2^1 c_1$ $1^4$	2 1	3 2 1	3 1 2 3 2 1	1 1 1 2 1 1 1 1	1 1 1 1		
5 $4^1$ $3^2$ $3^1 c_1$ $2^2 c_1$ $2^1 c_1^2$ $1^5$	2 1 1	5 4 3	4 3 5 5 4 3 1 2 1 1	3 2 3 5 3 4 3 3 1 2 1	1 1 1 2 1 2 1 1 1 1 1 1	1 1 1 1 1 1	
6 $5^1$ $4^2$ $4^1 c_1$ $3^3$ $3^2 c_1$ $3^1 c_1^2$ $2^3$ $2^2 c_1$ $2^1 c_1^2$ $1^6$	4 1 3	7 7 7 1 2 2 1	8 4 9 10 11 6 2 5 2 5 4 4 2 1 1 1	5 4 1 6 11 4 8 11 3 2 7 6 3 4 3 3 8 3 1 2 2 1 1 1	3 2 1 3 6 2 3 4 6 5 3 1 4 2 6 2 1 4 2 1 5 4 5 1 1 1 2 3 3 1 1 2 1 2	1 1 1 2 1 1 1 2 2 1 1 1 1 2 1 1 1 1 1 1 2 2 2 1 1 1 2 1 1 1 1 1 1 1 1	1 1 1 1 1 1 1 1 1 1



the example quoted at the beginning of this section. And from  $[c1] \tau [1^3] = [c1^3]$ , verifying the more general result (Mu55)

$$[c1] \tau [1^n] = [c1^n],$$

we get  $[31] \tau [1^3] = [1^4]$ .

Just as the out-plethysm  $[\mu] \overline{\circ} [\lambda]$  (Section 2.3) could be interpreted either as a plethysm for linear groups or as part of the  $a$ 'th out-power of  $[\mu]$ , where  $a = \#[\lambda]$ , a very similar second interpretation also exists for the in-plethysm. Let us define the  $a$ 'th in-power of  $[\mu]$  as the in-product of  $[\mu]$  by itself  $a$  times:

$$[\mu] \cdot [\mu] \cdot \dots \cdot [\mu].$$

The in-plethysm  $[\mu] \tau [\rho]$ , which we have discussed purely as a plethysm for symmetric groups  $H = S_k$ , also gives that part of the  $a$ 'th in-power of  $[\mu]$  which is of symmetry type  $[\rho]$  under the group  $S_a$ , provided  $[\mu] \tau [\rho]$  is weighted by  $\dim S_a[\lambda]$ . Of course  $a = \#[\lambda]$ . It follows that

$$[\mu] \cdot [\mu] \cdot \dots \cdot [\mu] = \sum_{[\rho]} [\mu] \tau [\rho] \dim S_a[\lambda]. \quad \underline{2.50}$$

The in-plethysms worked out as examples above allow us to verify that

$$\begin{aligned} [31] \cdot [31] \cdot [31] &= [31] \tau [3] + 2[31] \tau [21] + [31] \tau [1^3] \\ &= [4] + 4[31] + 3[21^2] + 2[2^2] + [1^4]. \end{aligned}$$

The in-product  $[31] \cdot [31] \cdot [31]$  is readily evaluated by 2.18 of Section 2.2, but it is sufficient here to give a dimensional check, using Table 2-1:

$$(3)^3 = 1 + 4 \times 3 + 3 \times 3 + 2 \times 2 + 1.$$

This closes the description of in-plethysm, the last YD operation to be presented in this thesis. The chapter ends with a general remark about the practical problem of calculating these operations in non-trivial cases.

We have emphasized that for both products and for both plethysms



there are an infinite number of different dimensional checks available for each special case. For in-products and for in-plethysms of type  $[c1] \tau [\rho]$ , this is because the same semi-general formula, expressed in  $[c\beta]$  notation, applies to all symmetric groups; for out-products and out-plethysms it is because the same explicit formula applies to all linear groups. This means that the dimensional checks actually determine the results of these operations and could be used to evaluate them in the absence of other procedures, since they provide any number of simultaneous linear equations to be solved for the coefficients of the YD's on the right hand side (assuming a sufficient number of these equations are linearly independent). In practice other methods will often provide partial answers and, given a more extensive table similar to Table 2.1, the dimensional checks will make it possible to identify the missing terms quickly by trial and error methods reminiscent of the calculations involved in the game of blackjack. The availability of electronic computers makes this more relevant: although most existing programs for solving systems of linear equations are not designed to take advantage of situations where both unknowns and coefficients are non-negative integers, programs can be written for this purpose as needed. Simplified procedures arising from new mathematical work would of course be even more valuable, particularly for plethysms.





## Chapter 3      MANY-NUCLEON FUNCTIONS

### 3.1    Many-Particle Functions, $S_k$ , $U_s$ , and $SU_s$

The mathematical techniques of the previous chapter had to be presented in considerable detail because no presentation suitable for our purposes seems to exist. Now that we have come to the physical application of group theory to many-particle wave functions, however, we may refer the reader to Hamermesh (H62) and to Elliott (E58, and especially E62) for a general discussion. Having stated the general ideas qualitatively in Section 1.1, we discuss only a few selected points in this section.

Let us look at the problem of  $k$  nucleons, each in one of  $s$  possible single-nucleon eigenstates. (In the LS-coupling approach the  $s$  eigenstates might refer to  $s$  possibilities for the space part of the wave function alone; the spin and isospin parts are considered separately, and when all are later put together we will see how the generalized Pauli Principle admits only space functions with not more than four of the  $k$  nucleons occupying each of the  $s$  eigenstates.) In general  $s$  and  $k$  may be any positive integers:  $s$  may be equal to, or greater or smaller than  $k$ .

Let the  $s$  eigenstates be denoted (temporarily) by letters  $a, b, c, d, \dots$ ; we now consider products of  $k$  such single-particle functions, and let the order of the  $k$  factors in such products designate which of the  $k$  particles is in which single-particle state. If for example  $k$  is 4, the four-nucleon eigenfunction  $adac$  means that the first and third nucleons are in the single-particle eigenstate designated by  $a$ , the second in the eigenstate  $d$  and the fourth in the eigenstate  $c$ .



It is obvious that we can form  $s^k$  different product functions in this way. For two particles ( $k=2$ ) in three possible single-particle states ( $a, b, c$ ), the  $s^{k=2}=9$  possible product functions are

$aa, ab, ac, ba, bb, bc, ca, cb, cc.$

For three particles in the three states  $a, b, c$ , we get  $s^{k=3}=27$  possibilities:

$aaa, aab, aac, aba, abb, abc, aca, acb, acc,$

$baa, bab, bac, bba, bbb, bbc, bca, bcb, bcc,$

$caa, cab, cac, cba, cbb, cbc, cca, ccb, ccc.$

As a third example, for three particles in two states there are  $s^{k=2}=8$  product functions:

$aaa, aab, aba, abb, baa, bab, bba, bbb.$

We will now classify these product functions according to their behaviour under various transformation groups, i.e. regroup them into linear combinations suitably chosen to form basis functions of the reps of these groups. Consider first the symmetric group  $S_k$ . A typical element of  $S_4$ , say, is the transposition  $(12)$  which interchanges objects labelled 1 and 2. Acting on the product function  $adac$ , this would interchange the first and second nucleons to produce

$$(12)adac = daac.$$

Similarly,  $(13)adac = adac;$

in this case the product function used as example happens to be invariant under that particular element of  $S_4$ .

If some standard choice is made for the form of the reps of  $S_k$ , there are definite techniques for constructing linear combinations that serve as basis functions. The results for the group  $S_2$  are well known: starting from the general form  $\alpha\beta$  for a product function one finds that the basis function belonging to the rep labelled by the YD  $[2]$  is the so-called symmetric



function, proportional to  $\alpha\beta + \beta\alpha$ , while for the rep  $[1^2]$  one gets the antisymmetric function, proportional to  $\alpha\beta - \beta\alpha$ . These are the only two reps of  $S_2$ , and since  $\dim S_2[2] = \dim S_2[1^2] = 1$ , there is only one basis function for each.

To classify with this scheme the nine product functions listed above for the example  $k=2$ ,  $s=3$ , we simply let each of the symbols  $\alpha$  and  $\beta$  in the functions  $\alpha\beta \pm \beta\alpha$  equal in turn each of the  $s$  symbols  $a, b, c$ , and then insert suitable normalization factors; some of the expressions obtained will vanish, and others, in general, will be omitted because they are not linearly independent of those already listed.

$[2]$	$\boxed{\begin{array}{ c c } \hline 1 & 2 \\ \hline \end{array}}$	$\alpha\beta + \beta\alpha$	$aa, bb, cc, (ab+ba)/\sqrt{2}, (ac+ca)/\sqrt{2},$ $(bc+cb)/\sqrt{2};$	6
$[1^2]$	$\boxed{\begin{array}{ c } \hline 1 \\ \hline 2 \\ \hline \end{array}}$	$\alpha\beta - \beta\alpha$	$(ab-ba)/\sqrt{2}, (ac-ca)/\sqrt{2}, (bc-cb)/\sqrt{2}.$	3

This process must of course yield exactly nine linearly independent functions since we are classifying the  $s^{k=3^2=9}$  products previously listed. The number of linearly independent basis functions obtained for each rep is listed at the right (6 and 3). The fact that these numbers are greater than one means that the classification as given so far is not sufficient to characterize the wave functions completely, and we will want to classify the same set of wave functions by means of other groups as well.

The corresponding results for two particles in two possible states ( $s^{k=2^2=4}$ ) are:

$[2]$	$\boxed{\begin{array}{ c c } \hline 1 & 2 \\ \hline \end{array}}$	$\alpha\beta + \beta\alpha$	$aa, bb, (ab+ba)/\sqrt{2};$	3
$[1^2]$	$\boxed{\begin{array}{ c } \hline 1 \\ \hline 2 \\ \hline \end{array}}$	$\alpha\beta - \beta\alpha$	$(ab-ba)/\sqrt{2}.$	1





For two particles and one possible state,  $s^{k=1^2}$  is one, and, trivially,

$[2]$	$\begin{array}{ c c } \hline 1 & 2 \\ \hline \end{array}$	$\alpha\beta + \beta\alpha$	$aa.$	1
$[1^2]$	$\begin{array}{ c } \hline 1 \\ \hline 2 \\ \hline \end{array}$	$\alpha\beta - \beta\alpha$		0

Before considering the complications that occur when  $\dim S_k[\lambda] > 1$ , let us use these simple two-particle examples to explain a somewhat tricky point. The statement is often made that the classification with respect to  $S_k$  as we have discussed it is at the same time a classification with respect to the unitary group  $U_s$ . This is true but requires some explanations, which are hardly ever hinted at (except briefly in Elliott's Mexican lectures (E62)).

The  $s$  single-particle eigenfunctions  $a, b, c, \dots$ , may be taken as the unit vectors in a linear vector space. If we apply to any vector of this space a non-singular linear transformation (i.e. multiply the column vector by an  $s \times s$  square matrix, an element of the group  $GL_s$ ) we obtain a new vector of the same space, i.e. another linear combination of the unit vectors  $a, b, c, \dots$ . In quantum mechanics the eigenfunctions form a linear vector space (more properly a ray space), with the added restriction that they must be normalized. By the principle of superposition any transformed vector is also a possible eigenfunction, provided the transformation has not disturbed the normalization. This normalization requirement restricts us to the subgroup  $U_s$  of  $s \times s$  unitary matrices instead of the more general  $GL_s$ . Finally, because of quantum mechanics' indifference to the phases of wave functions (we are really interested in a ray space rather than a linear vector space) we restrict our attention to the special-unimodular subgroup  $SU_s$  (of  $s \times s$  unitary matrices with determinant +1), since any transformation of  $U_s$ , because it can be factored into a transformation of  $SU_s$  and a





phase factor of modulus unity, leads to the same ray as the corresponding transformation of  $SU_3$ .

All this involved only the single-particle eigenfunctions  $a, b, c, \dots$ . It is natural however to define the effect of a transformation of  $SU_3$  on a many-particle function (which is a linear combination of products of single-particle functions) to be the result obtained when the transformation is applied to each of the single-particle functions involved. Consider an example with  $k=2$ ,  $s=3$ . Take the single-particle functions  $a, b, c$ , as unit vectors:

$$a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

A particular transformation of  $SU_3$  is

$$F = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Its effect on single-particle functions is found as follows:

$$\begin{aligned} Fa &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = b; & Fb &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = -a; \\ Fc &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = c; & F(a-b)/\sqrt{2} &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{1/2} \\ -\sqrt{1/2} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{1/2} \\ \sqrt{1/2} \\ 0 \end{bmatrix} = \sqrt{1/2}(a+b); \end{aligned}$$

and so on. Its effect on a two-particle function such as  $ab$  is

$$Fab = (Fa)(Fb) = (b)(-a) = -ba.$$

For some purposes it is useful to set up a Kronecker product formalism in which the example looks like this:



$$\begin{aligned}
 ab &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \\
 F &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \\
 \text{Therefore, } Fab &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = -ba.
 \end{aligned}$$

This illustrates that the  $k$ -nucleon functions (of which there are  $s^k$ , here  $3^2=9$ , as we have seen) form the basis functions of an  $s^k$ -dimensional representation of  $SU_s$ . To reduce this representation into reps of  $SU_s$  is to classify the  $k$ -nucleon functions, i.e. to construct linear combinations of them labelled by YD's of  $NR[\lambda] \leq s$  rows. But we have already constructed linear combinations labelled by YD's of  $\#[\lambda] = k$  boxes, when considering the group  $S_k$ . It is found that the linear combinations obtained now are precisely the ones listed earlier, and that moreover we had obtained exactly enough for each YD. Referring back to the example  $s=3$ ,  $k=2$ , we had found six linearly independent functions for  $[2]$  and three for  $[1^2]$ . These are the basis functions belonging to each row of the reps of  $SU_3$ : indeed (from Table 2.1)  $\dim U_3[2] = 6$  and  $\dim U_3[1^2] = 3$ . While these six functions, say, for  $[2]$  bore identical labels as a result of the  $S_2$  classification, they are now distinguished by means of the particular row of the rep  $[2]$  of  $SU_3$  to which each belongs.



In another language, the  $k$ -nucleon functions are tensors of degree  $k$  in  $s$  dimensions. To reduce the  $s^k$ -dimensional representation of  $SU_3$  mentioned, is to classify the tensors according to symmetry types  $[\lambda]$ , i.e. according to their behaviour under permutations of their  $k$  indices.

For all our  $k=2$  examples (we quoted  $s=3, 2, 1$ ) the fact that the numbers of linearly independent two-nucleon functions summed to  $s^k=s^2$  is now seen to reflect the identity

$$\dim U_s[2] + \dim U_s[1^2] = \frac{1}{2}s(s+1) + \frac{1}{2}s(s-1) = s^2. \quad \underline{3.1}$$

This is a special case (particularly simple because  $\dim S_2[\lambda] = 1$ ) of a more general result which we now illustrate.

For  $k>2$ , the expressions corresponding to  $\alpha\beta + \beta\alpha$  are more involved but are obtained by a well defined technique not reviewed here (see Section 7-11 of H62). One such expression is obtained for each Young Tableau (defined in Section 2.1) because we need a basis function for each of the  $\dim S_k[\lambda]$  rows of the rep of  $S_k$ . Then when we let each of  $\alpha, \beta$ , etc. equal in turn each of  $a, b, c, \dots$ , we obtain several linearly independent combinations of product functions, in fact exactly  $\dim U_s[\lambda]$  of them for each Young Tableau. We still obtain, necessarily,  $s^k$  functions, but their classification now reflects the more general identity

$$\sum_{[\lambda]} \dim S_k[\lambda] \dim U_s[\lambda] = s^k, \quad \underline{3.2}$$

where the sum is over all YD's with  $\#[\lambda] = k$  and  $NR[\lambda] \leq s$ . (Note that some of the rep classes of  $SU_3$  get counted more than once in this summation since we admit  $NR[\lambda] \leq s$ , not only  $NR[\lambda] \leq s-1$ .)

We have already enumerated  $s^k=27$  product functions for the case  $s=k=3$ . Their classification with respect to  $S_3$  is as follows. For each Young Tableau we write first (in Greek characters) the general form the linear combinations must have, then list (in Latin characters) all linearly





independent combinations obtained for  $s=3$ , by letting each of the  $k$  symbols  $\alpha, \beta, \gamma$  equal in turn each of the  $s$  symbols  $a, b, c$ .

$$[3] \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} \quad \alpha\beta\gamma + \alpha\gamma\beta + \beta\alpha\gamma + \gamma\beta\alpha + \gamma\alpha\beta + \beta\gamma\alpha; \quad 10$$

$$aaa, bbb, ccc,$$

$$(aab + aba + baa)/\sqrt{3}, \quad (aac + aca + caa)/\sqrt{3},$$

$$(bba + bab + abb)/\sqrt{3}, \quad (bbc + bcb + cbb)/\sqrt{3},$$

$$(cca + cac + acc)/\sqrt{3}, \quad (ccb + cbc + bcc)/\sqrt{3},$$

$$(abc + acb + bac + cba + cab + bca)/\sqrt{6};$$

$$[21] \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \quad \alpha\beta\gamma + \beta\alpha\gamma - \gamma\beta\alpha - \gamma\alpha\beta; \quad 8$$

$$(aab - baa)/\sqrt{2}, \quad (aac - caa)/\sqrt{2}, \quad (bba - abb)/\sqrt{2},$$

$$(bbc - cbb)/\sqrt{2}, \quad (cca - acc)/\sqrt{2}, \quad (ccb - bcc)/\sqrt{2},$$

$$(abc + bac - cba - cab)/\sqrt{4}, \quad (acb + cab - bca - bac)/\sqrt{4};$$

$$[21] \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \quad \alpha\beta\gamma - \beta\alpha\gamma + \gamma\beta\alpha - \beta\gamma\alpha; \quad 8$$

$$(baa - aba)/\sqrt{2}, \quad (caa - aca)/\sqrt{2}, \quad (abb - bab)/\sqrt{2},$$

$$(cbb - bcb)/\sqrt{2}, \quad (acc - cac)/\sqrt{2}, \quad (bcc - cbc)/\sqrt{2},$$

$$(abc - bac + cba + bca)/\sqrt{4}, \quad (acb - cab + bca - cba)/\sqrt{4};$$

$$[1^3] \quad \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \quad \alpha\beta\gamma - \alpha\gamma\beta - \beta\alpha\gamma + \beta\gamma\alpha + \gamma\alpha\beta - \gamma\beta\alpha; \quad 1$$

$$(abc - acb - bac + bca + cab - cba)/\sqrt{6}.$$

$$10 + 2 \times 8 + 1 = 27.$$

In cases such as this, where both  $\dim S_k[\lambda]$  and  $\dim U_s[\lambda]$  are greater than one for some  $[\lambda]$ , it is helpful to consider the basis functions as forming a rectangular array (E62) of  $\dim S_k[\lambda]$  columns and  $\dim U_s[\lambda]$  rows. This is illustrated for  $s = k = 3$ ,  $[\lambda] = [21]$ ,  $\dim S_2[21] = 2$ ,  $\dim U_3[21] = 8$ .



$\sqrt{\frac{1}{2}}(aab - baa)$	$\sqrt{\frac{1}{2}}(baa - aba)$
$\sqrt{\frac{1}{2}}(aac - caa)$	$\sqrt{\frac{1}{2}}(caa - aca)$
$\sqrt{\frac{1}{2}}(bba - abb)$	$\sqrt{\frac{1}{2}}(abb - bab)$
$\sqrt{\frac{1}{2}}(bbc - cbb)$	$\sqrt{\frac{1}{2}}(cbb - bcb)$
$\sqrt{\frac{1}{2}}(cca - acc)$	$\sqrt{\frac{1}{2}}(acc - cac)$
$\sqrt{\frac{1}{2}}(ccb - bcc)$	$\sqrt{\frac{1}{2}}(bcc - cbc)$
$\frac{1}{2}(abc + bac - cba - cab)$	$\frac{1}{2}(abc - bac + cba - bca)$
$\frac{1}{2}(acb + cab - bca - bac)$	$\frac{1}{2}(acb - cab + bca - cba)$

The ambiguous statement that these are basis functions for both  $S_k$  and  $SU_S$  simultaneously can now be understood more clearly. The  $\dim S_k[\lambda]$  functions in any one row of this array provide a basis for the rep  $[\lambda]$  of  $S_k$ , while the  $\dim U_S[\lambda]$  functions in any one column of this array provide a basis for the rep  $[\lambda]$  of  $SU_S$ .

Having clarified this particular point we will not discuss further the standard construction procedures for many-particle wave functions.

### 3.2 Shell Model Functions

The method of classification described in the preceding section is quite general and can be applied whether the single-particle functions involved refer to the space part of the wave function only, or to the spin part, the isospin part, the part describing spin and isospin together, or space and spin together, or the complete wave function.

The generalized Pauli Principle requires the complete wave function (space, spin, and isospin together) to be antisymmetric in all pairs of particles. In the notation developed here this means that only the complete



wave functions of symmetry type  $[1^k]$  are acceptable. If  $d$  represents the number of possible complete wave functions for a single nucleon, this means that the number of allowed  $k$ -nucleon functions is

$$\dim S_k[1^k] \dim U_d[1^k] = \binom{d}{k} = \frac{d!}{k! (d-k)!}. \quad \underline{3.3}$$

Of course this has a simple combinatorial interpretation: it is the number of ways of distributing the  $k$  nucleons among the  $d$  available states with due regard to the exclusion principle.

Suppose now that we apply these classification methods to the spin part of the wave function only. For one nucleon there are two states available:  $a$  and  $b$  represent spin up and spin down respectively. The unitary group involved is therefore  $SU_2$ , which is homomorphic to the ordinary rotation group  $R_3$ .

If we consider the isospin part of the wave function by itself we get exactly the same result:  $a$  and  $b$  now describe neutron and proton, considered as the two possible states of the nucleon. The group is again  $SU_2$ .

In the LS-coupling approach we consider separately, on one hand the space part of the wave function, and on the other hand the part describing spin and isospin together. For the latter part there are four possible states for a single nucleon, namely  $n\uparrow$ ,  $n\downarrow$ ,  $p\uparrow$ ,  $p\downarrow$ , in an obvious notation. The group is therefore  $SU_4$  and for  $k$  nucleons the reps are labelled by the YD's  $[\sigma]$  of  $k$  boxes and  $NR[\sigma] \leq 4$ . Actually, remembering the equivalence rules for special-unimodular groups, we see that the rep classes are labelled by triplets of numbers which we might choose in various ways; one possible choice is to label the rep classes by  $S_w = \frac{1}{2}(\sigma_1 + \sigma_2 + \sigma_3 - \sigma_4)$ ,  $T_w = \frac{1}{2}(\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4)$ , and  $Y_w = \frac{1}{2}(\sigma_1 - \sigma_2 - \sigma_3 + \sigma_4)$ . This notation provides the link with Wigner's original solution (W37) to the problem of enumerating the values of total spin  $S$  and total isospin  $T$  contained in the sets of  $k$ -





nucleon functions labelled by these rep classes  $[\sigma]$ . (Wigner's symbols  $S$  and  $T$  are here written  $S_w$  and  $T_w$  because they do not refer to total spin and isospin for which we reserve the symbols  $S$  and  $T$ .) In the language of Chapter 2 this problem amounts to the subduction  $SU_4 \rightarrow SU_2 \times SU_2$ , where the subgroup  $SU_2 \times SU_2$  is the direct product of a group referring to spin and another to isospin. The reduction of the subduced representation into reps of  $SU_2 \times SU_2$  produces a list of pairs of reps of  $SU_2$ , hence a list of values of total spin and total isospin. The general solution is given in approximately this language by Kretzschmar (K60a).

Consider now the space part of the wave function. Assume that there are  $s$  single-nucleon eigenstates available and that the  $k$ -nucleon space functions have been labelled by YD's  $[\lambda]$  as described in Section 3.1, referring to either  $SU_s$  or  $S_k$ . We therefore have reps of  $S_k$  labelled by  $[\lambda]$  on the space side and reps of  $S_k$  labelled by  $[\sigma]$  on the spin-isospin side, and we want to combine them to form complete wave functions belonging to the rep  $[1^k]$  of  $S_k$ , the only one allowed by the generalized Pauli Principle. The inner Kronecker product of reps of  $S_k$  corresponds to the in-product of YD's, so that this amounts to requiring  $[1^k]$  to appear in the in-product  $[\lambda] \cdot [\sigma]$ . But we mentioned in Section 2.2 that this occurs if and only if the factors are associate YD's:  $[\sigma] = [\tilde{\lambda}]$ . Admissible complete wave functions can be formed only by combining space functions labelled by a YD  $[\lambda]$  with spin-isospin functions labelled by its associate YD  $[\tilde{\lambda}]$ . (For the actual construction of these complete wave functions, see Trainor (T57) and references therein.)

Now we had  $NR[\sigma] \leq 4$  (because the spin-isospin group is  $SU_4$ ), so that the restriction  $[\sigma] = [\tilde{\lambda}]$  implies  $\lambda_1 \leq 4$ ; the YD's describing the space part of the wave functions are now restricted to at most four columns,





being associate to YD's of at most four rows. This is equivalent to saying that at most four nucleons, one in each of the available spin-isospin states  $n\uparrow$ ,  $n\downarrow$ ,  $p\uparrow$ ,  $p\downarrow$ , can have the same space function. Similarly the spin-isospin YD's  $[\sigma] = [\tilde{\chi}]$  are now restricted to at most  $s$  columns:  $\sigma_1 \leq s$ .

We now concentrate on the space part of the wave function and specialize to a definite model, the three-dimensional isotropic harmonic oscillator shell model in LS coupling. It is well known (E58, Me59) that there is a large degeneracy in the single-particle spectrum corresponding to this hamiltonian. The levels are evenly spaced in energy:

$$E_N = (N + 3/2)\hbar\omega, \quad N = 0, 1, 2, \dots, \quad \underline{3.4}$$

with  $\omega$  denoting the oscillation frequency of the corresponding classical problem. The  $N$ 'th level includes orbital angular momentum states with

$$\mathcal{L} = N, N-2, N-4, \dots, 0 \text{ or } 1, \quad \underline{3.5}$$

so that its degeneracy  $s$  (space part only) is given by

$$s = \Sigma(2\mathcal{L}+1) = \frac{1}{2}(N+1)(N+2) = 1, 3, 6, 10, \dots \quad \underline{3.6}$$

The magic numbers of this shell model, for neutrons or protons separately, are therefore

$$\begin{aligned} \Sigma_0^N(2s) &= (1/6)N(N+1)(2N+1) + (3/2)N(N+1) + 2(N+1) \\ &= (1/3)(N+1)(N+2)(N+3) = 2, 8, 20, 40, 70, 112, \dots \end{aligned} \quad \underline{3.7}$$

Since the experimental magic numbers are 2, 8, 20, (28), 50, 82, 126, the model is more or less relevant to experiment for the shells  $N=0, 1, 2$ , i.e. the  $s$ ,  $p$ , and  $sd$  shells filling at  $A=4, 16$ , and 40, and having space degeneracy  $s=1, 3$ , and 6 respectively.

One expects the ground state and the first few excited states of a nucleus in this mass region to belong to the ground configuration, with all but one shell either completely full or completely empty. (This expectation is not invariably confirmed by experiment.) We now designate



by  $s$  the degeneracy (space part only) of the one shell which is neither completely full nor completely empty, and by  $k$  the number of nucleons in that one shell. Considering only the space part of the wave function describing these  $k$  nucleons, we introduce a classification of the  $k$ -nucleon functions with respect to  $S_k$  and to  $SU_s$ . Thus for  $Ne^{20}$  the ground configuration is  $s^4 p^{12} (sd)^4$  and we concentrate on the  $k=4$  nucleons in the  $sd$  shell; the groups are  $S_4$  and  $SU_6$ .

For many purposes Coulomb forces may reasonably be neglected in this mass region so that we use the isospin formalism and think in terms of the total-nucleon magic numbers 4, 16, 40, ..., rather than the neutron or proton magic numbers 2, 8, 20, ... .

We have mentioned in Section 2.1 the equivalence for special-unimodular groups of the reps labelled by YD's that are the complements of each other. For example  $[43]$  and  $[41]$

			x
x	x	x	x

are equivalent for the group  $SU_3$ . This

makes possible a useful simplification in the classification procedure.

When  $k > 2s$ , i.e. when more than half of the  $4s$  possible states in the shell are occupied by nucleons, it is possible to discuss instead of this large number of nucleons the smaller number  $k' = 4s - k$  of holes remaining in the shell, and to classify these in terms of YD's of  $k'$  boxes.

With certain precautions all the results obtained for  $k$  particles are easily re-interpreted in terms of  $k$  holes so that we can avoid the larger YD's entirely in this way. The equivalence of YD's differing only by complete columns is also useful in simplifying certain calculations.

Several of the points mentioned in this section are illustrated in Table 3-1, which gives the classification of  $k$ -nucleon functions for the ground configuration  $s^4 p^k$ , in the  $p$  shell.



TABLE 3-1      Ground Configurations in the p Shell

A	k	$[\lambda]$	$[\tilde{\lambda}]$	$(2T+1, 2S+1)$	$[2] \overline{0}[\lambda], \text{ mesh } 2$	L	$[\lambda']$	k'	A'
4	0	$[0]$	$[0]$	(11)	$[0]$	0	$[4^3]$	12	16
5	1	$[1]$	$[1]$	(22)	$[2]$	1	$[4^2 3]$	11	15
6	2	$[2]$	$[1^2]$	( <u>31</u> )	$[4]+[2^2]$	2 0	$[4^2 2]$	10	14
		$[1^2]$	$[2]$	(11)(33)	$[31]$	1	$[4 3^2]$		
7	3	$[3]$	$[1^3]$	(22)	$[6]+[42]$	3 1	$[4^2 1]$	9	13
		$[21]$	$[21]$	(22)( <u>42</u> )	$[51]+[42]$	2 1	$[4 3 2]$		
		$[1^3]$	$[3]$	(22)(44)	$[3^2]$	0	$[3^3]$		
8	4	$[4]$	$[1^4]$	( <u>11</u> )	$[8]+[62]+[4^2]$	4 2 0	$[4^2]$	8	12
		$[31]$	$[21^2]$	( <u>31</u> )(33)	$[71]+[62]+[53]$	3 2 1	$[4 3 1]$		
		$[2^2]$	$[2^2]$	(11)(33)( <u>51</u> )	$[62]+[4^2]$	2 0	$[4 2^2]$		
		$[21^2]$	$[31]$	( <u>31</u> )(33)( <u>53</u> )	$[53]$	1	$[3^2 2]$		
9	5	$[41]$	$[21^3]$	(22)	$[91]+[82]+[73]+[64]$	4 3 2 1	$[4 3]$	7	11
		$[32]$	$[2^2 1]$	(22)( <u>42</u> )	$[82]+[73]+[64]$	3 2 1	$[4 2 1]$		
		$[31^2]$	$[31^2]$	(22)( <u>42</u> )(44)	$[73]+[5^2]$	2 0	$[3^2 1]$		
		$[2^2 1]$	$[32]$	(22)( <u>42</u> )(44)( <u>62</u> )	$[64]$	1	$[3 2^2]$		
10	6	$[42]$	$[2^2 1^2]$	( <u>31</u> )	$[10 2]+[9 3]+2[84]+[6^2]$	4 3 2^2 0			
		$[41^2]$	$[31^3]$	(11)(33)	$[93]+[75]$	3 1			
		$[3^2]$	$[2^3]$	(11)(33)	$[93]+[75]$	3 1			
		$[3 2 1]$	$[3 2 1]$	( <u>31</u> )( <u>33</u> )( <u>51</u> )( <u>53</u> )	$[84]+[75]$	2 1			
		$[2^3]$	$[3^2]$	( <u>31</u> )( <u>53</u> )( <u>71</u> )	$[6^2]$	0			





For each value of  $A$  or of  $k = A - 4$ , the table lists the allowed YD's  $[\lambda]$  for the space part of the wave function, subject to  $\#[\lambda] = k$ ,  $NR[\lambda] \leq 3$ , and  $\lambda_1 \leq 4$ . The groups are  $S_k$  and  $SU_3$ . The associate YD  $[\tilde{\lambda}]$  which describes the spin-isospin part of the wave function is then given, along with the allowed pairs of values of total isospin  $T$  and total spin  $S$  corresponding to it. The quantities  $T$  and  $S$  are indicated by the spectroscopic symbol  $(2T+1, 2S+1)$ . To save space, an underlined symbol means that the pair of values occurs in both orders: for example  $(\underline{31})$  means that both  $(31)$  and  $(13)$  occur, i.e. both  $T=1, S=0$ , and  $T=0, S=1$ . The quantities  $A$  and  $T$  determine the nuclides in which the corresponding states are expected to exist: a state labelled  $T$  should exist in the spectra of nuclides with atomic number  $Z$  given by

$$Z = \frac{1}{2}A - T, \frac{1}{2}A - T + 1, \dots, \frac{1}{2}A + T. \quad 3.8$$

The next two columns of Table 3.1 are explained in the next section. Finally the columns entitled  $[\lambda']$ ,  $k'$ ,  $A'$ , show how the classification just given for the first half of the shell ( $A = 4$  to  $10$ ) also applies to the second half ( $A = 11$  to  $16$ ). The YD  $[\lambda']$  shown in each line is defined by

$$\lambda'_i = 4 - \lambda_{4-i}, \quad i = 1, 2, 3. \quad 3.9$$

Thus  $[\lambda']$  is either the complement of  $[\lambda]$  for  $SU_3$ , exactly as defined in Section 2.2, or it is equivalent to that complement by the complete columns rule. (For a general shell of space degeneracy  $s$ , the definition would be

$$\lambda'_i = 4 - \lambda_{s+1-i}, \quad i = 1, 2, \dots, s.) \quad 3.10$$

In either case the rep  $[\lambda']$  of  $SU_3$  is equivalent to the rep  $[\lambda]$ . Moreover  $[\tilde{\lambda}']$  is equivalent to  $[\tilde{\lambda}]$  for  $SU_4$ , so that all the information in each line of the table applies to  $[\lambda']$  as well as to  $[\lambda]$ . But  $[\lambda']$  labels the space part of a  $k'$ -nucleon function, with  $k' = \#[\lambda'] = 4s - k = 12 - k$ , just as  $[\lambda]$  labelled the space part of a  $k$ -nucleon function. Thus it refers to the



system of mass number  $A' = k' + 4$ . Reading up the right hand side of the table one sees easily that every  $[\lambda]$  is accounted for in this way, that would have been allowed by the rules used in listing YD's on the left side of the table. Thus the complete classification for the second half of the shell is obtained from that for the first half.

In the  $A=10$  system, the identity of results for  $[41^2]$  and  $[3^2]$  is of course due to their equivalence: these YD's happen to be complements of each other for the group  $SU_3$ .

	x	x	x
	x	x	x

The classification which we have been applying to the harmonic oscillator shell model is really the supermultiplet classification and will be presented as such when its physical relevance is described in the next chapter. Having shown in Section 3.1 how the same classification can be considered either in terms of the symmetric group  $S_k$  or of the group  $SU_s$ , we will henceforth feel free to use the phrase 'supermultiplet classification' in connexion with either of these groups. But the main task of this chapter is to illustrate the application of YD techniques so that explanations need not interrupt the line of argument in later chapters. To this end we now use this same material in the next sections, to illustrate subclassification by plethysm, and the extension to configurations involving more than one shell.

### 3.3 Subclassification by Plethysm

We now continue to describe the classification procedure for the space part of  $k$ -nucleon functions in the  $p$ -shell ground configuration  $s^4 p^k$ ,



leading to the results given in Table 3-1. In particular we examine in this section the problem of listing the values of total orbital angular momentum  $L$  that occur in each of the sets of  $k$ -nucleon functions already labelled by YD's  $[\lambda]$ . This is entirely separate from the spin-isospin part of the problem.

As explained by Elliott (E58), the  $SU_s$  or  $S_k$  classification reflected the fact that the isotropic harmonic oscillator hamiltonian is invariant under these groups: all wave functions obtained from each other by transformations of  $SU_s$  (or of  $S_k$ ) are degenerate in energy for this hamiltonian. We now exploit the fact that the same hamiltonian is also invariant under ordinary space rotations, hence under transformations of the group  $SU_2$ , and introduce a subclassification which labels the  $k$ -nucleon space functions according to reps of  $SU_2$ , i.e. according to total orbital angular momentum  $L$ . From Chapter 2, these  $SU_2$  labels can be written as YD's  $[\nu]$ , with  $NR[\nu] \leq 2$  and  $L = \frac{1}{2}(\nu_1 - \nu_2)$ .

By hypothesis each of the  $k$  nucleons occupies a  $p$  state, i.e. has orbital angular momentum  $\ell = 1$ , so that  $L \leq k$ . The YD's  $[\lambda]$  of  $k$  boxes referred to the group  $SU_3$  (and also to  $S_k$ ); for a single nucleon we had  $[\lambda] = [1]$  with  $\dim U_3[1] = 3$ , the three basis functions being  $\phi(\ell, m) = \phi(1, 1)$ ,  $\phi(1, 0)$ ,  $\phi(1, -1)$  in an obvious notation. Since a single nucleon has  $L = 1$ , its  $SU_2$  label is therefore  $[\nu] = 2$ , satisfying  $L = \ell = \frac{1}{2}(\nu_1 - \nu_2) = 1$ , and  $\dim U_2[2] = 3$ , (to account for the three  $\phi(\ell, m)$ ). Note that  $\#[\nu] \neq k$ . In fact it will turn out that  $\#[\nu] = 2k$  for all  $k$  in this example.

The basic correspondence between the rep  $[\lambda] = [1]$  of  $SU_3$  and the rep  $[\nu] = [2]$  of  $SU_2$  had to come from the physics of the problem. Now that the correspondence is established for  $k=1$  however, we will see how the rest of the subclassification, i.e. the determination of such correspon-





dences for  $k > 1$ , can be obtained by mere YD manipulations without further appeal to physics.

Consider the case  $k=2$ . The two-nucleon functions are products of one-nucleon functions. Considering the one-nucleon functions as basis functions for the rep  $[2]$  of  $SU_2$ , we obtain the representation of  $SU_2$  based on the two-nucleon functions by taking the inner Kronecker product of the rep  $[2]$  of  $SU_2$  by itself. In YD language this becomes the out-product

$$[2] \circ [2] = [4] + [31] + [2^2],$$

corresponding to the Clebsch-Gordan series

$$D_1 \times D_1 = D_2 + D_1 + D_0.$$

The reader familiar with the quantum theory of angular momentum will here recall that the two-nucleon functions belonging to the reps  $D_2$  and  $D_0$  (or in general to any rep  $D_L$  with  $L$  even) of  $SU_2$  are symmetric under interchange of the nucleons, while those belonging to  $D_1$  (or to any odd  $L$ ) are antisymmetric. Thus the values  $L=0$  and  $2$  are to be associated with  $[\lambda] = [2]$  and  $L=1$  with  $[\lambda] = [1^2]$ , since  $[\lambda]$  describes the behaviour of the two-nucleon functions under the group  $S_2$  (as well as under  $SU_3$ ).

This subclassification which assigns  $L=0, 2$  to  $[\lambda] = [2]$  and  $L=1$  to  $[\lambda] = [1^2]$  will now be obtained from the YD formalism without explicit use of angular momentum theory. To do this we use the fact that  $[2]$  is the rep of  $SU_2$  corresponding to the rep  $[1]$  of  $SU_3$ , and take the out-plethysm  $[2] \overline{\circ} [\lambda]$ :

$$[2] \overline{\circ} [2] = [4] + [2^2],$$

$$[2] \overline{\circ} [1^2] = [31].$$

In the language of Section 2.3, the subduction  $G \rightarrow H$  that we are now considering is  $SU_3 \rightarrow SU_2$ . The rep  $\Gamma_{[\mu]}^H$  of the group  $H = SU_2$  is the one labelled  $[\mu] = [2]$ , because this is the one that corresponds to  $[\lambda] = [1]$





in the one-nucleon case. The dimension of  $\Gamma_{[\mu]}^H$  is

$$\dim U_2[2] = 3,$$

so that the matrices of  $\Gamma_{[\mu]}^H$  form a subgroup of the matrix group  $GL_3$ , or equivalently of  $SU_3$ . The dimensional checks for the out-plethysms quoted

$$\text{are} \quad \dim U_3[2] = \dim U_2[4] + \dim U_2[2^2],$$

$$6 = 5 + 1;$$

$$\text{and} \quad \dim U_3[1^2] = \dim U_2[31] = 3.$$

The meaning of the formula

$$[2]\overline{o}[2] + [2]\overline{o}[1^2] = [4] + [2^2] + [31] = [2]o[2]$$

should be clear from the description above, and provides a link with the second interpretation of out-plethysm presented in Section 2.3. That is, the out-plethysm  $[\mu]\overline{o}[\lambda]$  is that part of the  $a$ 'th out-power of  $[\mu]$  that is of symmetry type  $[\lambda]$  under the group  $S_a$ ,  $a$  being  $\#[\lambda]$ . In this example the out-plethysm  $[2]\overline{o}[\lambda]$  is that part of the second out-power of  $[2]$ ,

$$[2]o[2],$$

that is of symmetry type  $[\lambda]$  (i.e. symmetric for  $[\lambda] = [2]$  and antisymmetric for  $[\lambda] = [1^2]$ ) under the group  $S_2$  of permutations of the  $k=2$  nucleons.

For  $k=3$  we have, similarly (from Table 2.2),

$$[2]\overline{o}[3] = [6] + [42] + [2^3],$$

$$[2]\overline{o}[21] = [51] + [42] + [321],$$

$$[2]\overline{o}[1^3] = [41^2] + [3^2].$$

After sifting with mesh 2 (because  $H$  is  $SU_2$ ), and using  $L = \frac{1}{2}(\nu_1 - \nu_2)$ , we get the following subclassification.

$[\lambda]$	$[3]$	$[21]$	$[1^3]$
$L$	$3, 1$	$2, 1$	$0$
$\dim U_3[\lambda] = \Sigma (2L+1)$	$10 = 7 + 3,$	$8 = 5 + 3,$	$1 = 1.$



When we look at this example in the light of the second interpretation of out-plethysm, we notice something which did not show up in the  $k = 2$  example. The (sifted) third out-power of  $[2]$  is

$$[2] \circ [2] \circ [2] = [6] + 2[51] + 3[42] + [3^2].$$

To obtain this by adding (sifted) out-plethysms, the contribution of

$$[2] \overline{\circ} [21] = [51] + [42]$$

must be counted twice, because of the factor  $\dim S_a[\lambda]$ , here  $\dim S_3[21] = 2$ , in 2.35. What happens physically is easily understood: the linear combinations of 3-nucleon product functions form rectangular arrays as described at the end of Section 3.1. Either column of such an array is a basis for  $SU_3$ . The two functions in the same row of the array form a basis for  $S_3$ , i.e. differ only in the labelling of the nucleons and can not be distinguished physically from each other. Hence for all practical purposes we use simply the out-plethysm  $[2] \overline{\circ} [21]$  without counting it twice in the analysis of nuclear spectra.

For general  $k$  therefore the sifted out-plethysm  $[2] \overline{\circ} [\lambda]$  shown in the sixth column of Table 3.1 gives the values of  $L$  contained in the rep  $[\lambda]$  of  $SU_3$ . In the case of  $A = 10$ ,  $[\lambda] = [42]$ , the value  $L=2$  occurs twice.

This particular example of subduction by the out-plethysm  $[2] \overline{\circ} [\lambda]$  occurs with different physical interpretation on several occasions in  $SU_3$  theory.

### 3.4 Excited Configurations

We begin this section with an observation that has nothing to do with excited configurations directly but provides an analogy helpful in under-



standing their treatment.

In classifying the  $k$ -nucleon functions for the ground configuration  $s^4_p{}^k$ , we neglected the contribution of the four  $s$ -shell nucleons. Similarly in considering say the  $(sd)$ -shell ground configuration  $s^4_p{}^{12}(sd)^k$ , we would have neglected the sixteen nucleons of the inner core. It will be taken for granted here that there is physical justification for doing this, at least in a first approximation, because many interesting phenomena are well accounted for in this way.

But we now show that it is possible to take all  $A$  nucleons into account, and classify the  $A$ -nucleon space functions according to the symmetric group  $S_A$  instead of  $S_k$ , in such a way that the analysis of Section 3.2, which assigned  $[\lambda]$  to the corresponding spin-isospin part of the wave function, is not disturbed. Moreover we will point out in the next chapter that the physical relevance of this classification is also not disturbed.

Specializing again to  $s^4_p{}^k$ , we keep the symbol  $[\lambda]$ , with  $\#[\lambda]=k$ , for the  $k$ -nucleon part of the space function, and introduce the symbol  $[\alpha]$  for the four-nucleon space function describing the full  $s$  shell. Since there is only one possible space eigenstate in the  $s$  shell, the groups are  $S_4$  and  $SU_1$ , so that  $\#[\alpha] = 4$  and  $NR[\alpha] \leq 1$ . The only possibility is therefore  $[\alpha] = [4]$ , with  $\dim U_1[\alpha] = 1$  since  $SU_1$  is abelian.

To combine the rep  $[\lambda]$  of  $S_k$  and the rep  $[4]$  of  $S_4$  into a representation of  $S_A = S_{4+k}$ , we take the out-product  $[\lambda] \circ [4]$ . Referring to the explanations of Section 2.2, this shows the physical assumption we are making: permutations of  $S_A$  that interchange nucleons between the two shells are neglected. We are not really interested in  $S_A$  but only in its subgroup  $S_4 \times S_k$ .

Now we have seen in Section 3.2 that only those  $[\lambda]$  with at most





four columns are physically acceptable. For the same reason the YD's describing reps of  $S_A$  obtained from the out-product  $[\lambda] \circ [4]$  will be physically acceptable only if they have at most four columns. From the rules given in Section 2.2 for the graphical evaluation of out-products, the YD's constituting  $[\lambda] \circ [4]$  are obtained by adding to  $[\lambda]$  the four boxes of the YD  $[4]$ , all in different columns. These two requirements together imply that the only possible answer is the YD  $[\sigma]$  obtained from  $[\lambda]$  by adding one box in each of four columns. Therefore to each rep bearing an  $S_k$  label  $[\lambda]$  corresponds exactly one rep bearing the  $S_A$  label  $[\sigma] = [c\lambda]$ .

As an example consider the state  $[\lambda] = [42]$ , with  $k=6$ ,  $A=10$ .

The out-product is

$$[42] \circ [4] = [82] + [81^2] + [721] + [64] + [631] + [62^2] + [541] + [532] + [4^22],$$

but of all the YD's of  $A=10$  boxes appearing on the right hand side only  $[\sigma] = [c\lambda] = [4^22]$  is physically acceptable since all others have more than four columns.

$$\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \quad \circ \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline \end{array} \quad \rightarrow \quad \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & 1 & 1 \\ \hline & & 1 & 1 & \\ \hline \end{array}$$

Having established this  $S_A$  classification for the space part of the  $A$ -nucleon functions, we conclude as in Section 3.2 that the corresponding spin-isospin part of the  $A$ -nucleon function must be labelled by the associate YD  $[\tilde{\sigma}]$ , which indicates at the same time a rep of  $S_A$  and of the spin-isospin group  $SU_4$ . The purpose of the requirement that  $[\sigma]$  should have no more than four columns was of course to make its associate  $[\tilde{\sigma}]$  an acceptable rep of  $SU_4$ , with not more than four rows.

But since  $[\sigma]$  was just  $[\lambda]$  with an extra box in each of four columns,  $[\tilde{\sigma}]$  is just  $[\tilde{\lambda}]$  with an extra box in each of four rows, in other words  $[\tilde{\lambda}]$  with an extra column of four boxes.  $[\tilde{\sigma}]$  is therefore equivalent to  $[\tilde{\lambda}]$  for



the spin-isospin group  $SU_4$ , and its analysis into pairs of values of total spin  $S$  and total isospin  $T$  is the same as that of  $[\tilde{\lambda}]$ . The  $S_k$  and  $S_A$  classifications are thus equivalent in this respect. For  $[\lambda] = [42]$  and  $[\sigma] = [4^2 2]$ , the associates  $[\tilde{\lambda}] = [2^3 1]$  and  $[\tilde{\sigma}] = [3^3 2]$  are equivalent, and either procedure leads to the two possibilities  $T=1, S=0$ , and  $T=0, S=1$  indicated by the spectroscopic symbol (31) in Table 3.1.

The same correspondence between  $S_k$  and  $S_A$  classifications holds for the ground configurations in higher shells. Full shells of 4s nucleons always correspond to unique YD's of the form  $[4^s]$  for the space part:  $[\alpha] = [4]$  for the full s shell,  $[\beta] = [4^3]$  if the p shell is full,  $[\gamma] = [4^6]$  if the (sd) shell is full, and so on. The out-product of two or more of these is necessarily a single YD of the same form  $[4^t]$ , after the physically unacceptable YD's of more than four columns are rejected. Finally the out-product of  $[4^t]$  by the YD  $[\lambda]$  for the last shell always leads to a unique  $[\sigma]$  consisting of  $[\lambda]$  plus  $t$  extra rows of four boxes each. Again  $[\tilde{\sigma}]$  is equivalent to  $[\tilde{\lambda}]$  for  $SU_4$  by the complete columns rule.

The above discussion of out-products was phrased entirely in terms of the symmetric groups. Returning to the simple example  $[4]o[\lambda]$ , note that we interpreted it in terms of  $S_4$ ,  $S_k$ , and  $S_{4+k}$ . The dual interpretation of the out-product as the inner Kronecker product of reps of a single linear group is apparently lost here because  $[4]$  is a rep of  $SU_1$  and  $[\lambda]$  of  $SU_3$ . To restore a linear-group interpretation it is convenient to change to the associate YD's which describe the spin-isospin parts of the wave functions. It is not difficult to see from the graphical procedure for out-products, that the associate of the out-product of two YD's is the out-product of their associates (except for sifting). Thus the out-product  $[4]o[\lambda]$  considered above can be thought of as the associate of  $[1^4]o[\tilde{\lambda}]$ . But both



$[1^4]$  and  $[\tilde{\lambda}]$  refer to the same linear group, the spin-isospin group  $SU_4$ , so that their out-product has a legitimate dual interpretation. Indeed the restriction of  $[\sigma]$  to at most four columns now appears very naturally, as the usual requirement that the out-product  $[1^4] \circ [\tilde{\lambda}]$  must be sifted, in this case with mesh 4: this limits  $[\tilde{\sigma}]$  to four rows, hence  $[\sigma]$  to four columns. Of course we can go on taking out-products of the space YD's without invoking the associates in every instance, having understood once and for all from the arguments of this paragraph how the linear-group interpretation of these out-products can be restored as needed.

Having learned how to combine YD's referring to different shells, we now generalize this into a treatment of excited configurations. That is, we no longer require that all but one shell should be either completely full or completely empty. As an example the ground configuration of  $Ne^{20}$  is  $s^4 p^{12}(sd)^4$ ; the lowest excited configurations are  $s^4 p^{11}(sd)^5$  and  $s^4 p^{12}(sd)^3(pf)^1$ , both involving one extra oscillator quantum of energy  $\hbar\omega$  for the unperturbed harmonic oscillator hamiltonian. If we classify the space wave functions for the particles within each shell as in Section 3.2, we obtain a classification with respect to  $S_A$  by taking out-products just as we have done for ground configurations so far in this section.

Of course it is no longer true, in excited configurations, that all but one of the factors of this out-product are YD's of the form  $[4^s]$ . Hence it is no longer true that only one YD  $[\sigma]$  in the out-product is physically acceptable. As an illustration the various physically acceptable possibilities for the first three configurations of  $Ne^{20}$  are enumerated in Table 3.2. This example is used in later chapters.

This concludes the general exposition of YD techniques and of their use in the classification of many-particle eigenfunctions. From this point





on the concepts and techniques developed so far will be used not for the sake of illustrating them but more concisely to expound or develop certain points in the supermultiplet and the  $SU_3$  theories of nuclear structure.

TABLE 3-2  $S_A$  Classification for Space Functions in  $Ne^{20}$

configuration	$[\alpha]$ s	$[\beta]$ p	$[\gamma]$ (sd)	$[\delta]$ (pf)	$[\alpha] \circ [\beta] \circ [\gamma] \circ [\delta]$ ( $\leq 4$ columns)
$s^4 p^{12} (sd)^4$	$[4]$	$[4^3]$	$[4]$		$[4^5]$
20 $\hbar\omega$			$[31]$		$[4^4 31]$
ground			$[2^2]$		$[4^4 2^2]$
			$[21^2]$		$[4^4 21^2]$
			$[1^4]$		$[4^4 1^4]$
$s^4 p^{11} (sd)^5$	$[4]$	$[4^2 3]$	$[41]$		$[4^5] + [4^4 31]$
21 $\hbar\omega$			$[32]$		$[4^4 31] + [4^4 2^2] + [4^3 3^2 2]$
excited			$[31^2]$		$[4^4 31] + [4^4 21^2] + [4^3 3^2 1^2]$
			$[2^2 1]$		$[4^4 2^2] + [4^4 21^2] + [4^3 32^2 1]$
			$[21^3]$		$[4^4 21^2] + [4^4 1^4] + [4^3 321^3]$
			$[1^5]$		$[4^4 1^4] + [4^3 31^5]$
$s^4 p^{12} (sd)^3 (pf)^1$	$[4]$	$[4^3]$	$[3]$	$[1]$	$[4^5] + [4^4 31]$
21 $\hbar\omega$			$[21]$		$[4^4 31] + [4^4 2^2] + [4^4 21^2]$
excited			$[1^3]$		$[4^4 21^2] + [4^4 1^4]$





## Chapter 4      SUPERMULTIPLY THEORY

### 4.1    A Model for p-Shell Ground Configurations

The classification procedure which we have applied in Section 3.2 to the  $k$ -nucleon functions of a ground configuration, say  $s^4p^k$ , for the isotropic harmonic oscillator shell model in LS coupling, is a restatement in modernized language of the well known supermultiplet classification, which was developed and applied during the 'prehistoric' period of the shell model by Feenberg (F37, F37a), Wigner (F37, W37, W37a), Phillips (F37a), and Hund (Hu37). An elementary account of some of this material has been given by Sachs (S53).

It is shown in this series of papers that the classification of  $k$ -nucleon functions described in Section 3.2 is particularly convenient for treating a certain independent-particle model or family of models, in that these eigenfunctions produce an energy matrix diagonal in the interaction energies assumed.

The assumptions of the model are that the interactions between all pairs of nucleons are equal (charge independence of nuclear forces and neglect of Coulomb forces) and are independent of ordinary spin. We will be interested only in Wigner and/or Majorana central forces, and we assume that the radial dependence of these two types is the same. Actually several different sets of assumptions and several successive approximations are considered in each of these papers; we select those leading most simply to results relevant to the main theme of this thesis. Similar results are given for forces involving spin exchange but they will not be discussed here.



It is found under these assumptions that the contribution to the binding energy arising from interactions between the  $k$  particles in the  $p$  shell (ground configuration) can be expressed as a function of two parameters  $\mathcal{L}$  and  $\mathcal{K}$ :

$$E_p = \omega \mathcal{L} + \gamma \mathcal{K}. \quad \underline{4.1}$$

$\mathcal{L}$  and  $\mathcal{K}$  are integrals involving the radial dependence of the interaction assumed, along with single-particle functions for a pair of particles. For purposes of general discussion they can be treated as constants for the whole shell.

The coefficient  $\omega$  depends only on the quantum numbers  $[\lambda]$ , and can be determined by considering the limiting case of long-range forces (compared to the  $p$ -shell radius). For ordinary (i.e. Wigner) forces, in this limiting case,

$$\omega_w = \frac{1}{2}k(k-1); \quad \underline{4.2}$$

this is simply the number of pairs of particles being considered. For space exchange (i.e. Majorana) forces,

$$\omega_m = \frac{\frac{1}{2}k(k-1)\chi(T)}{\dim S_k[\lambda]} = \Xi. \quad \underline{4.3}$$

$\chi(T)$  is the character of a transposition in the rep  $[\lambda]$  of the group  $S_k$ , and  $\Xi$  is given in Table 2.1 for  $k = 1$  to 7.

To see this let  $\Psi_K, \Psi_\lambda$ , etc. be  $k$ -nucleon space eigenfunctions belonging to the same rep  $[\lambda]$  of  $S_k$ , so that the effect of a transposition of the  $i$ 'th and  $j$ 'th nucleons is

$$P_{ij} \Psi_K = \sum_{\lambda} D(ij)_{\lambda K} \Psi_{\lambda}, \quad \underline{4.4}$$

where  $D(ij)$  is the matrix representing this transposition in the rep  $[\lambda]$ .

If the Majorana interaction operator for the nucleon pair  $ij$  is  $\mathcal{J}(r_{ij})P_{ij}$ , we have



$$\begin{aligned}
 E_p &= V_{KK} = \langle \Psi_K | \sum_{i \neq j} g(r_{ij}) P_{ij} | \Psi_K \rangle \\
 &= \sum_{i \neq j} \sum_{\lambda} D(ij)_{\lambda K} \langle \Psi_K | g(r_{ij}) | \Psi_{\lambda} \rangle.
 \end{aligned}
 \tag{4.5}$$

In the long-range limit  $g(r_{ij})$  is considered constant over the range where the functions  $\Psi_{\lambda}$  are appreciable, so that by orthogonality,

$$\langle \Psi_K | g(r_{ij}) | \Psi_{\lambda} \rangle \approx \sigma_{K\lambda} \mathcal{L}, \tag{4.6}$$

$$\text{and} \quad E_p \approx \mathcal{L} \sum_{ij} D(ij)_{KK} = \frac{\frac{1}{2}k(k-1) \chi(T)}{\dim S_k[\lambda]} \mathcal{L} = \Xi \mathcal{L}, \tag{4.7}$$

as we claimed above. The term  $\gamma \chi$  vanishes in the long-range limit.

Unlike  $\omega = \Xi$ , the coefficient  $\gamma$  depends on the orbital angular momentum  $L$  as well as on  $[\lambda]$ , and its derivation is more complicated.

Now it can be shown that in general  $\mathcal{L} \geq 3\chi \geq 0$ . If we neglect the term  $\gamma \chi$  temporarily, we see that for each  $k$ , the binding energy increases with the quantity  $\Xi$ . We have assumed a hamiltonian independent of spin and isospin so that all values of  $S$  and  $T$  belonging to the same  $[\lambda]$ , as given in the fifth column of Table 3-1, are degenerate in energy for this model. We have given in Section 2.1 a standard order for listing the YD's of  $k$  boxes. This particular order was chosen for two reasons: it is easy to apply, and it coincides approximately, as Table 2-1 shows, with the order of decreasing  $\Xi$ . Since we used this standard order in listing the admissible  $[\lambda]$  for the ground configuration  $s^4 p^k$  in Table 3-2, the present model predicts that the order of states in the energy spectrum is as given in Table 3-2: e.g. for  $A=8$ , the ground and first few excited states should belong to  $[\lambda] = [4]$ , followed by  $[31]$ ,  $[2^2]$ , and  $[21^2]$  in order of increasing excitation.

Tables by Feenberg and Phillips (F37a), partly reproduced by Sachs





(S53), give explicitly the dependence of  $E_p$  on  $\mathcal{L}$ ,  $\mathcal{K}$ ,  $[\lambda]$ ,  $L$ ,  $T$ , and  $S$ , for Wigner, Majorana, Heisenberg, and Bartlett forces, for the lowest  $[\lambda]$  of each  $k$  in the  $p$ -shell ground configuration. As an example we present in the first five columns of Table 4-1 some of this information for the  $A=8$  system. The other columns will be explained later.

The first three columns list the states, and are obtained from Table 3-1. In the column for Majorana forces, we note that  $\omega = \Xi = 6$  and 2 respectively, and in the column for Wigner forces  $\omega = \frac{1}{2}k(k-1) = 6$  for all  $[\lambda]$ . The states listed would be followed by  $[\lambda] = [2^2]$  and  $[21^2]$ , for which  $\Xi$  is 0 and -2 respectively.

The model predicts that all states listed on the same line of Table 4-1 are degenerate in energy; these states may be said to form a super-multiplet, although the exact usage of this word varies somewhat.

A partial success of the model, well reviewed by Sachs (S53), is the correct prediction of most of the  $p$ -shell ground states. For example the ground state of  $\text{Be}^8$ , predicted to be  $^{11}\text{S}$  (first line of the table), i.e.  $T = S = L = 0$ , does have  $J = T = 0$  experimentally. Although this particular example is not very exciting, the model provides on the whole considerable evidence for the physical relevance of the classification procedure, despite the fact that the absolute binding energies are absolutely wrong.

The model also predicts, with some success, differences in binding energy between isobars of different  $T$ . For example in the  $A=8$  system the states of  $[\lambda] = [4]$  should occur only in  $\text{Be}^8$ , for which  $T = 0$ , whereas some of the states of  $[\lambda] = [31]$  have  $T = 1$  and should occur in  $\text{Li}^8$  and  $\text{B}^8$  also; in particular the ground states of  $\text{Li}^8$  and  $\text{B}^8$  should be the  $^{31}\text{P}$  or  $^{33}\text{P}$  of  $[31]$ .

Finally if we make the reasonable assumption that  $\mathcal{L}$  varies smoothly



TABLE 4-1    Low States for A = 8

$[\lambda]$	$\Xi$	$(2T+1, 2S+1)_L$	average potential energy		observed in $\text{Be}^8$		$(\lambda\mu)$	$B_0$	B
			Majorana	Wigner	energy	$J^\pi$			
[4]	6	$1^1\text{S}$	$6\mathcal{L} + 4\mathcal{K}$	$6\mathcal{L} + 4\mathcal{K}$	ground	$0^+$	(40)	56	56
		$1^1\text{D}$	$6\mathcal{L} + \mathcal{K}$	$6\mathcal{L} + \mathcal{K}$	2.90 Mev	$2^+$			47
		$1^1\text{G}$	$6\mathcal{L} - 6\mathcal{K}$	$6\mathcal{L} - 6\mathcal{K}$	11.7 Mev	$4^+$			26
[31]	2	$3^1\text{P}, 1^3\text{P}, 3^3\text{P}$	$2\mathcal{L} + 7\mathcal{K}$	$6\mathcal{L} - 5\mathcal{K}$			(21)	32	29
		$3^1\text{D}, 1^3\text{D}, 3^3\text{D}$	$2\mathcal{L} + 5\mathcal{K}$	$6\mathcal{L} - 7\mathcal{K}$					23
		$3^1\text{F}, 1^3\text{F}, 3^3\text{F}$	$2\mathcal{L} + 2\mathcal{K}$	$6\mathcal{L} - 10\mathcal{K}$					14



throughout the shell, we can try to correlate the fine structure (the kinks) in the curve of ground state binding energy vs  $A$ , with variations of  $\Xi$ . One can separate  $\Xi$  as follows,

$$\Xi = -k^2/8 + 2k + 5/2 - \Xi', \quad 4.8$$

into a part that varies smoothly with  $A$  throughout the shell, and a part  $\Xi'$

that has period 4 when expressed as a function of  $A$  for the ground states:

$$\Xi' = 5/2, 35/8, 5, 35/8, \text{ for the lowest admissible } [\lambda] \text{ in the systems}$$

$A = 4n, 4n+1, 4n+2, 4n+3$  respectively ( $n = 0, 1, 2, \dots$ ). Thus the model

predicts what we might describe as a partially magic character for the  $A=4n$

nuclei. This particular effect extends significantly beyond the  $p$  shell, both

theoretically and experimentally. Wigner (W37a) has expressed it by a term

$\Xi' \mathcal{L}(A)$  in a semi-empirical mass formula for the light nuclei, which has

been confirmed experimentally by Barkas (Bk39) and by Collins et al. (C52).

The empirical values  $\mathcal{L}(A)$  so obtained can then be used to predict spacings

between different  $[\lambda]$  in the excited spectra of single nuclei.

It should be noted that for shells higher than the  $p$  shell the effect

mentioned in the last paragraph is the only part that can be carried over with

some confidence from the  $p$ -shell treatment. It is not true for higher shells

that  $E_p = \omega \mathcal{L} + \gamma \mathcal{K}$ ; a much larger number of parameters would be required

instead of  $\mathcal{K}$ . While the  $\omega \mathcal{L}$  term, on which the discussion has concentrated

so far, does have some meaning for higher shells, the rest of this section

will be concerned with the  $\gamma \mathcal{K}$  term, and is strictly limited to the  $p$  shell.

In contrast to  $\omega$ , which depended on  $[\lambda]$  alone, the coefficient  $\gamma$

of the smaller parameter  $\mathcal{K}$  depends both on  $[\lambda]$  and on  $L$ . The various

admissible  $[\lambda]$  having been spaced out in the spectrum by the  $\omega \mathcal{L}$  term in

order of decreasing  $\Xi$ , the term  $\gamma \mathcal{K}$  then provides a splitting within each

rep  $[\lambda]$ . It was noted immediately (F37a) that the model described above



predicts an energy dependence proportional to  $L(L+1)$ : within each rep  $[\lambda]$ ,

$$\gamma \kappa = \gamma_0 \kappa - \frac{1}{2} L(L+1) \kappa. \quad 4.9$$

The first term  $\gamma_0 \kappa$  depends on  $[\lambda]$  and on the particular mixture of Wigner and Majorana forces considered, but not on  $L$ , whereas the second term depends only on  $L$ . Although the full significance of this in terms of the nuclear rotational model could not be appreciated of course in 1937, we would now express these facts as follows. Not only does the supermultiplet classification, in listing the allowable values of  $L$ , predict one or more rotational bands (cut off at some maximum  $L$ ) for each  $[\lambda]$ , as can be seen in Table 3-1, but the supermultiplet model of this section predicts for these states exactly the energy dependence

$$E_{\text{rot}} = \frac{\hbar^2}{2\mathcal{I}} L(L+1) \quad 4.10$$

typical of the rotational model in its simplest form (if  $L = J$ ), with moment of inertia  $\mathcal{I} = \hbar^2 / \kappa$ .

As an example we see in Table 4-1 that for  $A = 8$  the model predicts the rep  $[4]$  to be a band with  $L = 0, 2, 4$ : for either Majorana or Wigner forces the expressions  $\omega L + \gamma \kappa$  listed show differences of  $3\kappa$  between  $L=0$  and  $L=4$ , consistent with differences in the values of  $\frac{1}{2}L(L+1)$ , namely  $0, 1, 3, 6, 10, \dots$  for  $L = 0, 1, 2, 3, 4, \dots$  respectively. Similarly the rep  $[31]$  is a band with  $L = 1, 2, 3$  in which the differences of  $2\kappa$  between  $L=1$  and  $L=2$ , and of  $5\kappa$  between  $L=1$  and  $L=3$ , are consistent with  $\frac{1}{2}L(L+1)$ , (even though the term  $\gamma_0 \kappa$  is in this rep not the same for Wigner as for Majorana forces).

Thus the supermultiplet classification and the supermultiplet model have (in the  $p$  shell only) all the virtues that we claimed in Section 1.1 for the  $SU_3$  classification and the  $SU_3$  model (yet to be described): they obtain from independent-particle assumptions predictions typical of the rotational





model. The point is that the supermultiplet classification, as we pointed out in Section 3.2, is an  $SU_s$  classification as well as an  $S_k$  classification, hence is an  $SU_3$  classification in the case of the  $p$  shell (only) for which  $s$  is 3. We will see in the next chapter that such rotational model results are in fact typical of the  $SU_3$  classification and the main point of the work described there is to show how the  $SU_3$  classification can be introduced (as a subclassification of the  $SU_s$ ) in higher shells as well.

Despite the great interest of this connexion between models, the connexion with actual nuclei is fairly remote, since there seems to be only one convincing rotational band in actual  $p$ -shell nuclei. This is formed by the ground and first two excited states of  $Be^8$ , as listed (A59) in Table 4-1. The experimental moment of inertia  $\mathcal{I}$  is  $0.93 \hbar^2/\text{Mev}$ . Moreover the predicted energy dependence of the rotational model is  $J(J+1)$ , which coincides with  $L(L+1)$  only if  $S = 0$ , as in the  $Be^8$  example quoted. These points will be discussed further when the  $SU_3$  classification is introduced for its own sake in the next chapter.

To summarize, two points in this section have direct bearing on the  $SU_3$  classification. The first is that for all shells, the  $SU_s$  classification has some physical relevance and is useful in identifying low-lying states; because of this the  $SU_3$  classification will be introduced as a subclassification presupposing it. The second point is that, in the  $p$  shell only, the  $SU_s$  classification is automatically an  $SU_3$  classification, with all the properties that make  $SU_3$  classifications interesting, in particular the prediction of rotational bands; moreover the supermultiplet model predicts the typical rotational band energy spacings, just as the  $SU_3$  model of Chapter 5 will be seen to do.



## 4.2 Excited Configurations and the Supermultiplet Model

In the last section of Chapter 3 a method of extending the supermultiplet classification to excited configurations was proposed. It was shown in support of this method that for ground configurations the  $S_A$  classification obtained gives the same results as the previous  $S_k$  classification in one respect: the associate YD's classifying the spin-isospin A-nucleon functions are equivalent for  $SU_4$  to those previously given for k-nucleon functions, hence have the same sets of allowed values of T and S.

In this chapter we have introduced a model which predicts that for ground configurations, states belonging to various reps  $[\lambda]$  of  $S_k$  or of  $SU_s$  will be ordered in the excitation spectrum according to the quantity  $\Xi$ . (The other predictions of the model, being limited to the p shell, will not be of interest in this section.) We now show that the same predicted spectrum is obtained by ordering reps according to the quantities  $\Xi$  of the A-nucleon YD's obtained by taking out-products  $[\alpha]o[\beta]o\dots o[\lambda]$ , as in Section 3.4, of YD's representing each shell.

This is evident from the following example. In the ground configuration  $s^4p^{12}(sd)^4$  of  $Ne^{20}$ , the usual  $S_k$  classification orders states in the spectrum according to the values of  $\Xi$  belonging to the allowed YD's  $[\lambda]$  of  $k = 4$  boxes. The corresponding YD's

$$[\sigma] = [4]o[4^3]o[\lambda]$$

of  $A = 20$  boxes have already been obtained in the first part of Table 3-2. For comparison we list here the quantities belonging to both sets of YD's. Since the zero of energy is given no absolute significance in the model, the excitation energy predictions of the last column (below) are mere differences and are obtained just as readily from the  $\Xi_{[\sigma]}$  as from the  $\Xi_{[\lambda]}$ .



$[\lambda]$	$\Xi_{[\lambda]}$	$[\sigma]$	$\Xi_{[\sigma]}$	excitation
$[4]$	6	$[4^5]$	-10	0
$[31]$	2	$[4^4 31]$	-14	$4\mathcal{L}$
$[2^2]$	0	$[4^4 2^2]$	-16	$6\mathcal{L}$
$[21^2]$	-2	$[4^4 21^2]$	-18	$8\mathcal{L}$
$[1^4]$	-6	$[4^4 1^4]$	-22	$12\mathcal{L}$

From the physical assumptions of the model with Majorana forces,  $\Xi_{[\lambda]}$  is an approximation to the interaction energy contribution of all pairs formed from the  $k$  nucleons in the (sd) shell only, whereas  $\Xi_{[\sigma]}$  is the corresponding quantity for pairs formed from all  $A$  nucleons. Hence for a given configuration they differ only by an irrelevant constant.

To see this mathematically we use a convenient expression for  $\Xi$  established by Hund (Hu37):

$$\Xi = \frac{1}{2} \sum_i \lambda_i (\lambda_i - 2i + 1) = \frac{1}{2} \{ \lambda_1(\lambda_1 - 1) + \lambda_2(\lambda_2 - 3) + \lambda_3(\lambda_3 - 5) + \dots \}. \quad \underline{4.11}$$

The reader accustomed to thinking of each row of a YD  $[\lambda]$  as a subset of particles under interchanges of which the wave functions of the rep  $[\lambda]$  are symmetric, and of each column as a subset under interchanges of which they are antisymmetric, will recognize from Hund's expression that  $\Xi$  is the number of symmetrically coupled pairs minus the number of antisymmetrically coupled pairs:

$$\Xi = \frac{1}{2} \sum_i \lambda_i (\lambda_i - 2i + 1) = \sum_i \{ \frac{1}{2} \lambda_i (\lambda_i - 1) - (i - 1) \lambda_i \}.$$

The significance of this in terms of the derivation of  $E_p \approx \Xi \mathcal{L}$  for Majorana forces, given in Section 4.1, is clear. (It also becomes clear that

$$\Xi[\tilde{\lambda}] = -\Xi[\lambda]^c)$$





Now we showed in Section 3.4 that the  $[\sigma]$  corresponding to a given  $[\lambda]$  for the ground configuration is uniquely determined and of the form

$$[\sigma] = [4^t, \lambda_1, \lambda_2, \dots, \lambda_i, \dots]; \quad 4.12$$

$$\text{i.e. } \sigma_1 = \sigma_2 = \dots = \sigma_t = 4, \quad \text{and } \sigma_{t+i} = \lambda_i.$$

$$\begin{aligned} \text{Hence } \Xi[\sigma] &= \frac{1}{2} \sum_i \sigma_i (\sigma_i - 2i + 1) = \frac{1}{2} \sum_{i=1}^t 4(4 - 2i + 1) + \frac{1}{2} \sum_{i>t} \lambda_{i-t} (\lambda_{i-t} - 2i + 1) \\ &= \frac{1}{2} \{ 20t - 4t(t+1) \} + \frac{1}{2} \sum_j \lambda_j (\lambda_j - 2j + 1) - tk \quad 4.13 \\ &= (8-k)t - 2t^2 + \Xi[\lambda]. \end{aligned}$$

For the  $\text{Ne}^{20}$  example,  $t = k = 4$ ; the 'irrelevant constant' is therefore  $16\mathcal{L}$ , in agreement with the numerical values tabulated above.

What we have done so far in this section is to show that for ground configurations the supermultiplet model is just as meaningful when formulated in terms of the A-nucleon YD's obtained as out-products of YD's representing each shell, as when formulated in terms of the k-nucleon YD's for the one shell that is neither completely full nor completely empty. This suggests that for excited configurations also the supermultiplet model should be meaningful if the classification described in Section 3.4 is adopted. For example we have obtained in Table 3.2 the various possible A-nucleon YD's for the two lowest excited configurations of  $\text{Ne}^{20}$ . We can expect that under the assumptions of the supermultiplet model the excitation energy of the corresponding A-nucleon states will include a contribution  $-\Xi[\sigma]\mathcal{L}$ .

In particular the YD  $[\sigma]$  with the highest value of  $\Xi$  for each configuration, which should therefore have lowest energy, will be called the leading YD. The leading YD generally has as few rows as possible; or, equivalently, it has as many of its boxes as possible arranged in complete rows of four boxes. The leading YD for each of the three configurations of



$\text{Ne}^{20}$  shown in Table 3.2 is  $[4^4]$ , with  $\Xi = -10$ , as can be seen from the values of  $\Xi$  tabulated in this section.

For brevity a shell will be called active if it is neither completely full nor completely empty. To illustrate this usage we restate a definition already given: the ground configuration by definition has at most one active shell (the ground configuration of a closed-shell nucleus has none).

If the ground configuration has  $k$  nucleons in its active shell, consider the excited configuration formed by promoting one of these  $k$  nucleons to the next higher shell, e.g.  $s^4p^{12}(\text{sd})^3(\text{pf})^1$  in  $\text{Ne}^{20}$ . For the equally-spaced single-particle levels of the unperturbed harmonic oscillator spectrum this means one extra quantum of energy, and no other excited configuration can have lower energy so that we can call this the first excited configuration (even if it is first ex aequo as in  $\text{Ne}^{20}$ , where both  $s^4p^{11}(\text{sd})^5$  and  $s^4p^{12}(\text{sd})^3(\text{pf})^1$  have 21 quanta).

We now establish a fact needed later: the leading YD of the first excited configuration is always the same as the leading YD of the ground configuration.

To see this recall that the only possible YD for an inactive shell is  $[4^8]$  or  $[0]$ , and the out-product for any number of inactive shells is a unique YD of the form  $[4^t]$ ; hence the ground configuration has a unique YD

$$[\sigma] = [4^t] \circ [\lambda] = [4^t, \lambda_1, \lambda_2, \dots]$$

if  $[\lambda]$  describes its  $k$ -nucleon active shell. Now for the first excited configuration we have the out-product

$$[4^t] \circ [\lambda'] \circ [1]$$

with any  $[\lambda']$  satisfying  $\#[\lambda'] = \#[\lambda] - 1$ . From the graphical procedure of Section 2.2 for the evaluation of out-products,  $[4^t] \circ [\lambda'] \circ [1]$  is the sum of all YD's formed by adding one box to  $[4^t] \circ [\lambda']$ ; in particular if we choose



a  $[\lambda']$  formed by removing one box from the leading  $[\lambda]$  of the ground configuration, we can always replace that box in the same location. Hence the leading YD of the ground configuration necessarily occurs in the first excited configuration (and in fact in all configurations). To show that it is in fact the leading YD of the first excited configuration also, we need only recall that the leading YD of the ground configuration has by construction the largest  $\Xi$  of all YD's of  $A$  boxes with  $\sigma_1 \leq 4$ : its boxes are crowded into as few rows as possible, with at most one row having less than four boxes, thus maximizing the positive and minimizing the negative terms in the expression

$$\Xi = \frac{1}{2} \{ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \dots - \lambda_1 - 3\lambda_2 - 5\lambda_3 - \dots \}.$$

Hence the leading YD in the first excited configuration is also the leading YD in the ground configuration.

From this it follows that the position in the spectrum, of the lowest state belonging to an excited configuration is not affected by the  $\Xi$  dependence the supermultiplet model predicts, since it has the same  $\Xi$  as the ground state. In particular the position of the lowest state of abnormal (i.e. negative) parity in  $\text{Ne}^{20}$  (from  $s^4p^{12}(sd)^3(pf)^1$ ) with respect to the ground state is not affected by the term  $\Xi \mathcal{L}$  since they both belong to  $[4^5]$  with  $\Xi = -10$ . The supermultiplet model affects in no way the original prediction that this state should be located one oscillator quantum above the ground state, under the assumptions of the model.

When applying this statement to actual nuclei, it must be verified that the leading excited-configuration state so predicted is not spurious, in a sense to be described in Section 6.1. This verification is carried through in that section for the  $\text{Ne}^{20}$  example.

Finally it should be mentioned that Motz and Feenberg (Mt38) have applied the supermultiplet model to excited configurations of the systems





$4 \leq A \leq 16$ . They do not state how they obtain their allowed YD's for excited configurations, and they list explicit results for  $A = 16$  only. The method of this section will reproduce these results except that it does not allow  $[4^2 3^2 1^2]$  in the configuration  $(1s)^4(2p)^{10}(2s)^2$  as Motz and Feenberg do. (They do not consider the levels  $2s$  and  $3d$  to be degenerate, and the present attempt to reproduce their results of course makes the same assumption.) It is not clear whether this disagreement indicates a genuine difference in physical assumptions between the two methods.

To summarize, we have given in this section an extension of the supermultiplet formalism to excited configurations, in a way which preserves the physical relevance of the supermultiplet model. We have verified that the supermultiplet model does not modify the prediction of the unperturbed harmonic oscillator model in one particular point, namely the position of the leading excited-configuration state relative to the ground state in the energy spectrum. By contrast we will see at the end of the next chapter that the  $SU_3$  model, in many respects similar to the supermultiplet model, does predict a different energy for that level.





## Chapter 5 THE $SU_3$ CLASSIFICATION AND MODEL

### 5.1 The $SU_3$ Classification in Higher Shells

We have seen in the previous chapter that the supermultiplet classification of  $k$ -nucleon functions according to reps of the group  $SU_3$  has some particularly interesting properties when the single-particle space degeneracy  $s$  happens to equal 3, i.e. in the  $p$  shell. The ensuing  $SU_3$  classification groups states of orbital angular momentum  $L$  into bands similar to those of the rotational model, and the supermultiplet model gives these states an energy spacing proportional to  $L(L+1)$ .

Elliott (E58, 58a) has shown that these properties are characteristic of the group  $SU_3$  and has shown how an  $SU_3$  classification can be introduced into the higher shells as well, so that similar predictions, of interest because of the connexion they provide between the independent-particle approach and the collective approach, can be made also in these higher shells. In particular an  $SU_3$  model providing an energy spacing proportional to  $L(L+1)$  has also been devised for higher shells.

Elliott's  $SU_3$  classification procedure is based on the fact that the three-dimensional isotropic harmonic oscillator single-particle hamiltonian is invariant under a group of transformations whose structure is that of  $SU_3$ . After the position and momentum coordinates  $\underline{r}$  and  $\underline{p}$  are redefined in the usual way (E58, Me59) to absorb proportionality constants, the hamiltonian reads

$$H = \frac{1}{2} (r_x^2 + r_y^2 + r_z^2 + p_x^2 + p_y^2 + p_z^2). \quad \underline{5.1}$$

The reason why it has such a large symmetry group ( $SU_3$  is an 8-parameter group), or equivalently such a large degeneracy, is apparent from the form



of that expression: all six quantities  $r_i$  and  $p_i$  appear on the same footing. In any isotropic hamiltonian the fact that the three quantities  $r_x, r_y, r_z$  appear symmetrically on one hand, and the three quantities  $p_x, p_y, p_z$  appear symmetrically on the other hand, is expressed by the symmetry group  $SU_2$ . In exactly the same way the greater symmetry between all six quantities is expressed by the much larger symmetry group  $SU_3$ , of which  $SU_2$  is a subgroup.

These symmetries are conveniently formulated (E58) in terms of the usual creation and annihilation operators

$$\begin{aligned}\underline{a}^* &= \sqrt{\frac{1}{2}} (\underline{r} - i\underline{p}), \\ \underline{a} &= \sqrt{\frac{1}{2}} (\underline{r} + i\underline{p}).\end{aligned}\tag{5.2}$$

Consider the complex three-dimensional linear vector space whose unit vectors are the creation operators  $a_i^*$ . We will be interested in linear transformations  $T$  acting in that space. Now the hamiltonian can be expressed as  $H = \frac{1}{2} \sum_{i=x,y,z} (a_i a_i^* + a_i^* a_i)$ . Moreover it is well known that the  $r$ -quanta eigenfunctions of  $H$ , i.e. those belonging to the eigenvalue  $E_r = (r + 3/2) \hbar \omega$ , can all be expressed as  $P|0\rangle$ , where the ket  $|0\rangle$  represents the ground state (with only the zero-point energy  $3/2 \hbar \omega$ ), and  $P$  is a homogeneous polynomial of degree  $r$  in the three creation operators  $a_i^*$ . We will find it useful, for brevity, to speak of the eigenfunctions themselves as homogeneous polynomials, without mentioning the ket  $|0\rangle$ .

If we define as usual the effect of transformations  $T$  on  $H$  or  $P$  to mean the result obtained by applying  $T$  to each of the  $a_i^*$  appearing therein, it is easily seen that the transformations  $T$  under which  $H$  is invariant, or equivalently the transformations  $T$  that transform eigenfunctions  $P$  into other eigenfunctions  $P$  of the same energy  $r$ , form precisely the group  $SU_3$ .

This correspondence, expressing the  $r$ -quanta eigenfunctions as



homogeneous polynomials of degree  $r$  in three 'single-quantum' quantities  $a_i^*$ , is the key to a very important parallel between the  $SU_3$  classification now being introduced and the  $SU_5$  classification explained in Section 3.2. To the  $k$  nucleons of Section 3.2 correspond the  $r$  quanta. To the  $s$  possible single-nucleon eigenfunctions correspond the three possible single-quantum eigenfunctions represented by  $a_i^*$ ,  $i = x, y, z$ . To the  $k$ -nucleon eigenfunctions considered as homogeneous polynomials of degree  $k$  in  $s$  variables, or tensors of degree  $k$  in  $s$  dimensions, correspond the  $r$ -quanta eigenfunctions considered as homogeneous polynomials of degree  $r$  in three variables, or tensors of degree  $r$  in three dimensions. To the YD's  $[\lambda]$  of  $k$  boxes and  $\leq s$  rows correspond YD's  $[\rho]$  of  $r$  boxes and  $\leq 3$  rows. And to the permutation group  $S_k$  corresponds the permutation group  $S_r$ .

We are thus led to classify the eigenfunctions into reps of  $SU_3$  labelled by YD's  $[\rho] = [\rho_1, \rho_2, \rho_3]$  of at most three rows. Remembering that reps labelled by YD's differing only by complete columns of three boxes are equivalent for  $SU_3$ , we can denote the rep classes more concisely by a pair of non-negative integers

$$\lambda = \rho_1 - \rho_2, \quad \mu = \rho_2 - \rho_3, \quad \underline{5.3}$$

which we will always write  $(\lambda\mu)$ , in parentheses rather than square brackets. Remembering also that YD's complementing each other (as defined in Section 2.1) are equivalent for  $SU_3$ , we see that  $(\lambda\mu)$  and  $(\mu\lambda)$  are equivalent, so that the rep classes of  $SU_3$  are in one-one correspondence with the symbols  $(\lambda\mu)$  such that, say,  $\lambda \geq \mu$ . (In practice one often removes complete columns but keeps both  $(\lambda\mu)$  and  $(\mu\lambda)$  as they arise.)

In this notation we might rewrite an equation of Section 2.1 as

$$\dim U_3[\rho] = \frac{1}{2}(\lambda+1)(\mu+1)(\lambda+\mu+2). \quad \underline{5.4}$$

Now any value of  $(\lambda\mu)$  can occur in general for a system consisting





of any number of nucleons. The symbol  $(\lambda\mu)$  (or the symbol  $[\rho]$ ) does not in any way determine either  $A$  or  $k$ . Moreover  $[\rho]$ , in contrast to the supermultiplet symbol  $[\lambda]$ , is not restricted to four columns, since its associate  $[\tilde{\rho}]$  has no particular significance in terms of spin-isospin functions.

If this classification is to be useful in conjunction with the  $[\lambda]$  and  $L$  classifications already discussed, one must show that it does not disturb the earlier classifications, i.e. that the linear combinations it prescribes can be constructed without mixing terms belonging to different values of  $[\lambda]$  and  $L$ . Elliott has in effect done this by showing that  $SU_3$  is a subgroup of  $SU_8$  and that it includes the ordinary rotational group  $SU_2$  as a subgroup. Of course this does not mean only that these groups are subgroups of one another in their abstract structure, which is trivially true: he has shown this subgroup relationship to exist in the particular set of transformations acting on the wavefunctions represented by homogeneous polynomials of creation operators.

While Elliott, originally interested only in ground configurations, develops the  $SU_3$  classification in conjunction with the  $S_k$  rather than the  $S_A$  version of the supermultiplet classification, (i.e. in terms of the  $k$  nucleons of the active shell only rather than all  $A$  nucleons), it is not necessary to proceed in this way. If one does proceed in Elliott's way, the  $r$  quanta mentioned above will be only those attributed to the  $k$  particles in the active shell, so that  $r = kN$  if the active shell is the one in which single-particle states have energy

$$E_N = (N + 3/2)\hbar\omega. \quad \underline{5.5}$$

The fact that the  $SU_8$ ,  $SU_3$  and  $SU_2$  classifications all involve  $YD$ 's makes it possible to find their interrelationships concisely, with the techniques



of Chapter 2, thus taking advantage of calculations that have been done once and for all instead of having to repeat them in each particular physical context.

In particular the calculation of the reps  $[\rho]$ , or equivalently  $(\lambda\mu)$ , of  $SU_3$ , which are present in each of the reps  $[\lambda]$  of  $SU_s$  is a straightforward application of the technique of subduction by out-plethysm (Section 2.2) which we have already used and explained in Section 3.3 in connexion with a different subduction.

As an example consider the (sd) shell, for which  $N = 2$ ,  $s = 6$ . The subduction is  $SU_6 \rightarrow SU_3$ . To find into what reps  $[\rho]$  of  $SU_3$  the general rep  $[\lambda]$  of  $SU_6$  breaks down we consider first the case of  $k=1$ , with  $[\lambda] = [1]$ . A single particle in this shell is responsible for two quanta; we conclude that  $\#[\rho] = 2$ . This still leaves two possibilities:  $[2]$  or  $[1^2]$ .

To choose between these it is instructive to consider the dual interpretation of the  $SU_3$  classification as an  $S_r$  (in this case  $S_2$ ) classification. This implies a labelling of the two quanta, which has the same formal significance as the labelling of indistinguishable nucleons in a many-nucleon wave function. The reps  $[2]$  and  $[1^2]$  of  $S_2$  are symmetric or antisymmetric respectively under interchange of the quanta. Interchange of the quanta means simply interchange of the order of the two symbols  $a_i^*$  occurring in each term of the homogeneous polynomials of degree two which represent the two-quantum wave function. But it is well known (Me59) that all creation operators commute. Hence  $[\rho] = [2]$  and not  $[1^2]$ . In confirmation of this,  $\dim U_3[2] = 6 = \dim U_6[1]$ .

Knowing that  $[\rho] = [2]$  corresponds to  $[\lambda] = [1]$ , we can now find the values of  $[\rho]$  corresponding to the YD's  $[\lambda]$  for  $k>1$  without further appeal to physics: they are given by the out-plethysm  $[2] \overline{\circ} [\lambda]$ , sifted with mesh 3 because  $H$  is  $SU_3$  in this application.



In particular we list below the reps of  $SU_3$  found to occur in the leading rep of  $S_4$  or of  $SU_6$  for  $A = 20$ ,  $k = 4$ , ground configuration.

$$\begin{aligned} [\lambda] &= [4], & \dim U_6[4] &= 126; \\ [2]\overline{0}[\lambda] &= [8] + [62] + [4^2] + [42^2]; \\ \sum \dim U_3[\rho] &= 45 + 60 + 15 + 6 = 126; \\ (\lambda\mu): & (80) + (42) + (04) + (20). \end{aligned}$$

The reader may have noticed that we are using here the same out-plethysm data which we used in Section 3.3, but getting some completely different information out of them. For instance the out-plethysm  $[2]\overline{0}[4]$  used in Table 3-1 for the subduction  $SU_3 \rightarrow SU_2$  told us that the values  $L = 4, 2, 0$  occurred in the rep  $[4]$  of  $SU_3$  in  $Be^8$ . Now the same out-plethysm, used for  $SU_6 \rightarrow SU_3$ , provides the reps of  $SU_3$  present in the rep  $[4]$  of  $SU_6$  in  $Ne^{20}$ . The sifting, interpretation, and dimensional checks are different, but the same entry in the out-plethysm Table 2-2, calculated once and for all, holds the solution of these two apparently very different problems, and of many others. This illustrates the power of YD techniques in such applications and provides one reason why it is useful to define operations acting on the YD's themselves rather than on the reps of particular groups.

We have seen that the reps of  $SU_3$  corresponding to each rep  $[\lambda]$  of  $SU_S$  were obtained, in the (sd) shell for which  $N = 2$ , from the out-plethysm  $[2]\overline{0}[\lambda]$ , sifted with mesh 3. Exactly the same arguments hold in all shells. For the general shell labelled  $N$ , the  $k$  nucleons are responsible for  $r = kN$  quanta. If  $k = 1$ , the rep  $[\lambda] = [1]$  corresponds to the single-row YD  $[\rho] = [N]$ , the completely symmetric rep of  $S_r$ , since all creation operators commute. Hence for general  $k$  in that shell the subduction  $SU_S \rightarrow SU_3$  is found from the out-plethysm  $[N]\overline{0}[\lambda]$  sifted with mesh 3.





There are two trivial cases. In the  $p$  shell,  $N = 1$ . As we have seen, the  $SU_s$  classification is automatically an  $SU_3$  classification in that shell, so we merely set  $[\rho] = [\lambda]$ . This is consistent with the out-plethysm formula  $[1]\overline{\circ}[\lambda] = [\lambda]$ . The other trivial case occurs for the  $s$  shell,  $N = 0$ . Here we can not say of course that  $SU_3$  is a subgroup of  $SU_s$ , since  $s = 1$ . But the formalism could be rephrased to avoid trouble. Trivially,  $[\rho] = [0]\overline{\circ}[\lambda] = [0]$ , and the only rep of  $SU_3$  possible in this shell is  $(\lambda\mu) = (00)$ , of dimension one. (The subgroup structure  $SU_3 \supset SU_2 \supset SU_s = SU_1$  is just as satisfactory as the more usual  $SU_s \supset SU_3 \supset SU_2$ ,  $s \geq 3$ , since either relationship implies that none of the three classifications (supermultiplet,  $SU_3$ , and orbital angular momentum) need mix into the same rep of one group, functions that another has assigned to different reps of another group.)

Following Elliott, we have thus shown in principle how the  $k$ -nucleon functions for the ground configuration in any shell of the LS-coupling isotropic harmonic oscillator model can be classified according to reps of  $SU_3$ . Explicit results are tabulated in Elliott's first paper (E58) for parts of the (sd) and (pf) shells.

## 5.2 The $SU_3$ Model

The next step is to determine which values of the total orbital angular momentum  $L$  occur in each of the reps  $(\lambda\mu)$  of  $SU_3$ . This is exactly the problem discussed in Section 3.3, but is no longer restricted to the  $p$  shell. The subduction  $SU_3 \rightarrow SU_2$  is obtained from the out-plethysm  $[2]\overline{\circ}[\rho]$ , sifted with mesh 2 because  $H$  is the rotation group  $SU_2$ .

It is found that the results can be systematized as follows (E58). If





we always write  $(\lambda\mu)$  with  $\lambda \geq \mu$ , the reps of  $SU_2$  labelled by  $L$  that occur in  $(\lambda\mu)$  are given by

$$L = K, K+1, K+2, \dots, K+\lambda \quad \text{if } K > 0,$$

and by 
$$L = \lambda, \lambda-2, \lambda-4, \dots, 1 \text{ or } 0 \quad \text{if } K = 0, \quad \underline{5.6}$$

where 
$$K = \mu, \mu-2, \mu-4, \dots, 1 \text{ or } 0.$$

We refer to Table 3-1 for examples. The leading rep for  $A = 8$  is  $[\rho] = [\lambda] = [4]$ , i.e.  $(\lambda\mu) = (40)$ ; there  $K=0$  is the only possibility, with  $L = 4, 2, 0$ . The leading rep for  $A = 10$  is  $[\rho] = [\lambda] = [42]$ , or  $(\lambda\mu) = (22)$ ; in this case two values of  $K$  are allowed:  $K=2$ , with  $L = 2, 3, 4$ , and  $K = 0$  with  $L = 2, 0$ .

Neglecting temporarily the distinction between the total orbital angular momentum  $L$  and the total angular momentum  $J$  (orbital plus spin), we note the strong resemblance between these results and those of the simplest version of the rotational model, which assumes the nucleus to have cylindrical but not spherical symmetry. The quantum number  $K$  is then interpreted as the projection of  $L$  ( $J$  in the rotational model) on the axis of cylindrical symmetry. Thus each value of  $K$  represents an intrinsic state on which is superposed a band of  $L$  values including all integers  $\geq K$  (except for  $K=0$ , in which case either even values of  $L$  only or odd values of  $L$  only are allowed, depending on the parity).

Now in the rotational model all the states of a band can be expressed as integrals involving the intrinsic state of the band and the well known eigenfunctions of a symmetric top; this is well explained in an appendix to Moszkowski's review article (Mz57). Elliott (E58a) has shown that the same relationship holds for the states of the  $SU_3$  classification, thus explaining their property of forming rotational bands.

The connexion between  $SU_3$  eigenfunctions and rotational states



becomes even more impressive when we consider the following specific model. Moszkowski (Mz58) has presented physical arguments suggesting a residual interaction proportional to a so-called quadrupole-quadrupole operator  $Q^2$  (described also by Bargmann and Moshinsky (B60) and Elliott (E62)). If we assume a many-particle hamiltonian of the form

$$\mathcal{H} = H - Q^2/\mathcal{Q}, \quad 5.7$$

where  $H$  is the unperturbed harmonic oscillator many-particle hamiltonian and  $\mathcal{Q}$  is a proportionality constant, it is found that the  $SU_3$  eigenfunctions labelled by  $(\lambda\mu)$ ,  $K$ , and  $L$ , are still eigenfunctions of  $\mathcal{H}$  as well as of  $H$ , with eigenvalue

$$\begin{aligned} (r + 3/2)\hbar\omega - \frac{2\hbar^2}{3\mathcal{Q}} (\lambda^2 + 3\lambda + \lambda\mu + 3\mu + \mu^2) + \frac{\hbar^2}{2\mathcal{Q}} L(L+1) \\ = (r + 3/2)\hbar\omega - \frac{\hbar^2 B}{3\mathcal{Q}}, \end{aligned} \quad 5.8$$

where  $B = B_0 - (3/2) L(L+1)$ ,

and  $B_0 = 2(\lambda^2 + 3\lambda + \lambda\mu + 3\mu + \mu^2)$ .

(These definitions make both  $B_0$  and  $B$  integers.)  $B_0$  determines the relative positions of different reps  $(\lambda\mu)$  of  $SU_3$  in the energy spectrum, while the term  $B - B_0 = -(3/2) L(L+1)$  spaces out different states  $L$  within the same rep  $(\lambda\mu)$  in the manner typical of the rotational model, with the proportionality constant  $\mathcal{Q}$  serving as moment of inertia.

Consider for example the rotational band we have already discussed in  $Be^8$  (Table 4-1). The quantities  $B_0 = 56$  and  $B$  have been indicated in the last columns of Table 4-1. This band is thus accounted for equally well by the supermultiplet model of Section 4.1 or by the  $SU_3$  model of this section (or of course by the rotational model itself). The predictions of the  $SU_3$  and supermultiplet models are actually identical if we identify the quantity  $\mathcal{K}$  of the supermultiplet model with  $\hbar^2/\mathcal{Q}$ , where  $\mathcal{Q}$  is the proportionality constant



of the  $SU_3$  model.

Now suppose that for a particular nucleus, the parameter  $\mathcal{Q}$  of the  $SU_3$  model is fixed by comparison with the experimental energy spacings within a rotational band. The model then predicts the same moment of inertia for all bands in that nucleus. But because of the form of the eigenvalue expression 5.8 it also predicts a definite value for the energy term  $\hbar^2 B_0 / 3\mathcal{Q}$ , which fixes the order and spacing in the spectrum, of the various reps  $(\lambda\mu)$ . The term  $\hbar^2 B_0 / 3\mathcal{Q}$  thus plays a role analogous to that of the term  $\omega\mathcal{L}$  or  $\Xi\mathcal{L}$  in the supermultiplet model.

For the  $Be^8$  example, the last column of Table 4-1 shows that the P state of the rep (21) should lie slightly lower in the spectrum than the G state of (40). This is not observed experimentally since only the three predicted states of (40) can be identified with confidence in the present experimental spectrum, and they seem to be the three lowest. We will see later that in  $Ne^{20}$  such predictions of the relative position of different reps  $(\lambda\mu)$ , using the parameter  $\mathcal{Q}$  fixed from the spacings within each  $(\lambda\mu)$ , are in qualitative agreement with experiment.

For the p-shell ground configurations we can make the supermultiplet and the  $SU_3$  models predict identical energy differences between reps by setting  $\mathcal{L} = 3\hbar^2/\mathcal{Q} = 3\mathcal{K}$ . We refer again to Table 4-1 for an example. We want to study the energy difference between the two reps  $[\lambda] = [4]$  and  $[31]$ , or equivalently between  $(\lambda\mu) = (40)$  and  $(21)$ . To eliminate the L dependence already described, compare the D states occurring in each of the reps. The supermultiplet model predicts this energy difference to be  $4\mathcal{L} - 4\mathcal{K}$  for pure Majorana forces, or  $8\mathcal{K}$  for pure Wigner forces, while the  $SU_3$  model predicts a difference of  $24\hbar^2/3\mathcal{Q}$  (the number 24 is the difference between the values of  $B_0$  for  $(\lambda\mu) = (40)$  and  $(21)$ ). The correspondence





between models is independent of the particular mixture of Majorana and Wigner forces assumed in the supermultiplet model. Moreover the same assumption  $\mathcal{L} = 3\mathcal{K} = 3\hbar^2/\mathcal{Q}$  will make the predictions of the two models identical for the two leading reps of all p-shell nuclei, except for an additive constant which is of no significance.

Alternatively, if we are interested only in the  $B_0$  term (i.e. the L-independent energy difference between reps  $(\lambda\mu)$ , not within each rep), we can also obtain agreement between models for all reps of the p-shell ground configuration by a different set of assumptions. If we restrict ourselves to pure Majorana forces and set  $\mathcal{L} = 2\hbar^2/\mathcal{Q}$  and  $\mathcal{K} = 0$ , the L-independent energy difference between (40) and (21) in  $\text{Be}^8$  (same example as above, Table 4-1), is now simply  $4\mathcal{L}$  on the supermultiplet model, compared to  $24\hbar^2/3\mathcal{Q}$  on the  $\text{SU}_3$  model. In fact we prove below that in general  $(\Delta B_0) = 6(\Delta \Xi)$ , if  $(\Delta B_0)$  and  $(\Delta \Xi)$  represent the differences between the quantities  $B_0$  and between the quantities  $\Xi$  respectively for any two reps of the same configuration  $p^k$ . Hence  $\hbar^2(\Delta B_0)/3\mathcal{Q} = \mathcal{L}(\Delta \Xi)$  provided  $\mathcal{L} = 2\hbar^2/\mathcal{Q}$ .

The proof is as follows. We assume for simplicity that  $\rho_3 = 0$ .

Then  $\rho_1 = \lambda + \mu$  and  $\rho_2 = \mu$ , so that we obtain from 4.11

$$\Xi = \frac{1}{2}\{\rho_1(\rho_1 - 1) + \rho_2(\rho_2 - 3)\} = \frac{1}{2}(\lambda^2 + 2\lambda\mu + 2\mu^2 - \lambda - 4\mu); \quad \underline{5.9}$$

$$\begin{aligned} \text{hence } 6\Xi - B_0 &= 3(\lambda^2 + 2\lambda\mu + 2\mu^2 - \lambda - 4\mu) - 2(\lambda^2 + 3\lambda + \lambda\mu + 3\mu + \mu^2) \\ &= (\lambda + 2\mu)^2 - 9(\lambda + 2\mu) = (\rho_1 + \rho_2)^2 - 9(\rho_1 + \rho_2) = r^2 - \frac{5.10}{9r}. \end{aligned}$$

The difference  $6\Xi - B_0$  is thus a function only of  $r$  or of  $k=r$ , the number of quanta or of particles in the active p shell, hence is constant within the spectrum of each nucleus, and  $(\Delta B_0) = 6(\Delta \Xi)$ , Q.E.D.

From the agreement described above, between the predictions of the supermultiplet model with special assumptions and those of the  $\text{SU}_3$  model, for p-shell ground configurations, it follows that any evidence



supporting the physical relevance of the supermultiplet model in that shell can also be claimed to support the physical relevance of the  $SU_3$  model. We have mentioned in Section 4.1 that there is in fact some qualitative evidence to that effect. Elliott (E58) has also established a strong resemblance between  $SU_3$  eigenfunctions and some shell-model eigenfunctions adjusted to fit experimental spectra as a result of previous calculations, in a few nuclei at the beginning of the (sd) shell. This supports the physical relevance of the  $SU_3$  model in that mass region, which is interesting because a few nuclei around  $A \approx 23$  are known to be amenable to the rotational model.

The main limitation of the  $SU_3$  model, as well as of the version of the supermultiplet model presented here, is the neglect of spin-dependent forces. The interesting connexion between the  $SU_3$  model and the rotational model remains somewhat artificial as long as it involves  $L$  instead of  $J$ . Strictly speaking this limitation restricts the applicability of the connexion to the nuclei whose low states have  $S = 0$ , hence  $J = L$ , such as  $Be^8$  and  $Ne^{20}$ .

To summarize, three main points have been discussed in this section. One is that the  $SU_3$  model, for the ground configuration in any shell, arranges states within each rep  $(\lambda\mu)$  into bands with the energy spacings typical of the rotational model. The second is that in so doing the model predicts the energy spacings between different reps  $(\lambda\mu)$ , these spacings being determined by the same quantity  $\mathcal{Q}$  which appears as a moment of inertia within each band. And the third point is that in the p-shell ground configuration only, where the supermultiplet model also makes such predictions, the predictions of the two models can be made to coincide by special assumptions about the parameters of the supermultiplet model.



### 5.3 Excited Configurations and the $SU_3$ Model; $Ne^{20}$

We have considered in Section 3.4 a method for extending to excited configurations a YD classification of many-particle functions. The idea is to apply the usual classification scheme to the particles in each of the shells separately, then to combine the YD's representing each of the shells by the out-product operation. It is found that inactive shells contribute only trivially to such out-products but non-trivial out-products are obtained when two or more shells are active in a given configuration. If a model is known to make the classification physically relevant or otherwise interesting in ground configurations, one can attempt to apply the model to excited configurations, using the above procedure. In Sections 3.4 and 4.2 we have followed this program for the supermultiplet classification and established a few points about the supermultiplet model for excited configurations.

Because of the parallel between classifications (stressed in Section 5.1), the corresponding program for  $SU_3$  can be carried through without difficulty, leading to an attempt to apply the  $SU_3$  model to excited configurations. This is essentially the content of the present section.

In a way the meaning of these manipulations is clearer for the  $SU_3$  classification than it was for the supermultiplet classification. The latter involved reps of  $SU_s$ , with a different value of  $s$  for each shell, so that we had to think in terms of the associate YD's describing spin-isospin functions when we tried to visualize the out-product operation in its usual role as the inner Kronecker product of two reps of the same linear group. No such complication occurs for the  $SU_3$  classification (in which the associate YD's have no particular significance): the YD  $[\rho]$  describing each shell labels a rep of the same group  $SU_3$ , and the out-product is also a representation of





$SU_3$ . The sifting, with mesh 3, is straightforward and there is no limitation on number of columns.

We digress to note that the theory of holes (mentioned in Section 3.2), which essentially reduces the treatment of the second half of each shell to that of the first half, applies to the  $SU_3$  classification just as it did to the supermultiplet classification. If  $k$  particles in a shell are equivalent to  $4s-k$  holes, where  $4s$  is the maximum number of particles that can be put into the shell, it is also true that  $r = kN$  quanta attributed to the  $k$  particles are equivalent to  $4sN - r$  'quantum holes' since a full shell contains  $4sN$  quanta. Hence the  $r$ -quanta functions in the second half of the shell can be labelled by YD's  $[\rho]$  with  $\#[\rho] = 4sN - r$ , which are the complements of the YD's of  $r$  boxes for  $SU_3$ . The complement of  $(\lambda\mu)$  for  $SU_3$  is  $(\mu\lambda)$ . Since  $s = \frac{1}{2}(N+1)(N+2)$  from 3.6,  $4sN$  is necessarily divisible by three and a full shell is labelled by  $[\rho] = [f^3]$ , with  $f = 4sN/3$ ;  $[f^3]$  is equivalent to  $[0]$  for  $SU_3$  (either by the complete columns rule or by the complements rule) so that full shells, like empty shells, are labelled by  $(00)$  or  $[0]$ , which acts as an identity element for the out-product operation. This conclusion allows us to take out-products of the YD's  $[\rho]$  for active shells only, when considering excited configurations.

We now consider what is probably the most interesting application of the  $SU_3$  classification and model to an actual nucleus. (Much of the discussion concerns the ground configuration and can be considered as a detailed illustration of the material of the last section. It is convenient to discuss it here however, along with the material concerning excited configurations specifically.) The spectrum of  $Ne^{20}$  has been subjected to a concerted experimental investigation by the Chalk River group (Lh61), who observed five rotational bands. Elliott briefly reported to the Manchester Conference





(Mk61) an  $SU_3$  identification of these bands, and Levinson and Meshkov reported further work attempting to improve agreement with experiment by a generalization of  $SU_3$  theory (which we will not discuss).

The experimental spectrum is pictured in Lh61. We present in Table 5-1 and in this section a numerical comparison between these data and the predictions of the  $SU_3$  model. It is not our purpose to obtain the best possible fit by introducing further residual interactions with adjustable parameters, using  $SU_3$  theory only as a guide in choosing zero-order states for the perturbation calculation. Instead we consider the  $SU_3$  model eigenfunctions without further mixing, with eigenvalues

$$(r + 3/2)\hbar\omega - \frac{\hbar^2 B}{3\mathcal{I}}.$$

While this produces only order of magnitude agreement with experiment it shows what relative accuracy is obtained for the various types of predictions implied by the model.

Three of the five observed rotational bands are assigned to the ground configuration  $s^4 p^{12} (sd)^4$ , and one to each of two excited configurations. We discuss first the ground configuration. We have already seen that the leading rep  $[\lambda]$  of  $SU_6$  for  $Ne^{20}$  is  $[4]$ , or equivalently  $[4^5]$  (Section 4.2), and that it contains the reps of  $SU_3$  labelled  $(\lambda\mu) = (80), (42), (04)$ , and  $(20)$  (Section 5.1). We have defined the leading  $[\lambda]$  as the rep of  $SU_6$  favoured by the supermultiplet model because it has the largest value of  $\Xi$ . Similarly we now define the leading  $(\lambda\mu)$  or  $[\rho]$  as the rep of  $SU_3$  favoured by the  $SU_3$  model because it has the largest value of  $B_0$ . Of the four reps  $(\lambda\mu)$  mentioned,  $(80)$  is the leading rep with  $B_0 = 176$ , followed by  $(42)$  with 92,  $(04)$  with 56, and  $(20)$  with 20. The leading state of the configuration is the one which is leading on both counts, e.g. the state which bears an  $SU_6$  label  $[4]$  and an  $SU_3$  label  $(80)$  in this example. It is therefore



TABLE 5-1  $\text{Ne}^{20}$ : Experimental Data and  $\text{SU}_3$  Predictions

configuration	$(\lambda\mu)$	K	$J^\pi$	excitation energy E (Mev)	B	intra-band $\frac{B}{E^2}$ ( $\text{MeV}^{-1}$ )
$s^4p^{12}(\text{sd})^4$	(80)	0	$0^+$	ground	176	<u>av. 2.32</u>
			$2^+$	1.63	167	1.84
			$4^+$	4.25	146	2.35
			$(6^+)$	7.6	113	2.76
	(42)	0	$0^+$	6.75	92	<u>av. 4.26</u>
			$2^+$	7.45	83	4.29
			$4^+$	9.11	62	4.24
	(04)	0	$0^+$	7.22	56	<u>av. 4.69</u>
			$2^+$	7.86	47	4.69
	(82)	2	$2^-$	4.97	*219	<u>av. 3.69</u>
			$3^-$	5.63	*210	4.55
			$(4^-)$	7.02	*198	3.41
			$(5^-)$	8.84	*183	3.10
$s^4p^{11}(\text{sd})^5$	(90)	0	$1^-$	5.80	*213	<u>av. 3.60</u>
$s^4p^{12}_{\text{or}}(\text{sd})^3(\text{pf})^1$			$3^-$	7.19	*198	3.60



encouraging to note in the experimental spectrum that the lowest band can be labelled (80) and two others (42) and (04). All of these belong to  $[\lambda] = [4]$ . From 5.6, only  $K=0$  occurs in (80) and (04). However both  $K=0$  and  $K=2$  bands should occur in (42), and the latter is not observed. We must assume that its energy is raised in some manner not accounted for by the  $SU_3$  model, since the eigenvalue expression 5.8 is independent of  $K$ .

The first three columns of Table 5-1 identify the bands as in the discussion above, and the next two give the observed  $J$  (which is equal to  $L$  because  $S$  is zero), parity  $\pi$ , and excitation energy. The next column gives the quantity  $B$  which represents the predictions of the  $SU_3$  model. The symbols  $*$  are a reminder that for excited configurations an extra  $\hbar\omega$  is also involved so that the  $B$  values marked  $*$  are not to be compared directly with those not so marked; this is discussed below.

In the next column we list the value of the parameter  $\mathcal{Q}$  (the moment of inertia) which makes the  $SU_3$  prediction of column  $B$  agree with the experimental excitation energy. More precisely, we compare the predicted and observed excitation energies of each state above the lowest state of its own band, namely  $-\hbar^2(\Delta B)/3\mathcal{Q}$  and  $\Delta E$ , and obtain  $\mathcal{Q}$  by equating these two quantities. For example the state labelled (42),  $4^+$  has  $\Delta E = 9.11 - 6.75 = 2.36$  Mev, and  $\Delta B = 62 - 92 = -30$ , hence

$$\mathcal{Q} = \frac{-\hbar^2 \Delta B}{3 \Delta E} = 4.24 \hbar^2/\text{Mev}.$$

In the spaces opposite the lowest state of each band in the last column is given the average of these quantities  $\mathcal{Q}$  for each band, i.e. the moment of inertia for each band. A perfectly rotational  $J(J+1)$  spectrum would imply identical entries  $\mathcal{Q}$  for all states in the same band; moreover, perfect agreement between the  $SU_3$  model and experiment would imply identical entries in the entire column, since the model predicts the same moment of





inertia for all bands.

The entries are in fact far from identical but they do agree somewhat better than in order of magnitude, the extremes being in a ratio of 5:2. Otherwise we would not be attempting a discussion in terms of the rotational and  $SU_3$  models. The band averages seem to increase somewhat with excitation energy, an effect for which the  $SU_3$  model makes no provision.

Now the point where the  $SU_3$  model goes beyond the rotational model is that it predicts the inter-rep energy spacings should be governed by the same parameter  $\mathcal{Q}$  as the intra-band spacings already considered. We test this by calculating the quantities  $\mathcal{Q}$  that produce agreement between predicted and observed values, for the energy differences between the lowest states in different reps. The results are

$$(80) \rightarrow (42): \quad \mathcal{Q} = \frac{-\hbar^2 \Delta B}{3 \Delta E} = \frac{\hbar^2 (176 - 92)}{3(6.75 - 0) \text{ Mev}} = 4.15 \hbar^2 / \text{Mev};$$

$$(80) \rightarrow (04): \quad \mathcal{Q} = \frac{\hbar^2 (176 - 56)}{3(7.22 - 0) \text{ Mev}} = 5.54 \hbar^2 / \text{Mev};$$

and similarly in the excited configurations,

$$(82) \rightarrow (90): \quad \mathcal{Q} = \frac{\hbar^2 (219 - 213)}{3(5.80 - 4.97) \text{ Mev}} = 2.41 \hbar^2 / \text{Mev}$$

Again, perfect agreement between the  $SU_3$  model and experiment would imply a unique value for all the quantities  $\mathcal{Q}$  calculated above, i.e. both intra-band and inter-rep together. The overall average is  $\mathcal{Q} = 3.60 \hbar^2 / \text{Mev}$ . For comparison the moment of inertia of a rigid sphere of radius  $1.2 \sqrt[3]{20} \text{ fm}$  is  $2.0 \hbar^2 / \text{Mev}$ .

We now consider the excited configurations. Referring to Table 3-2,  $s^4 p^{11}(\text{sd})^5$  and  $s^4 p^{12}(\text{sd})^3(\text{pf})^1$  are the lowest excited configurations, each being one oscillator quantum above the ground configuration. We have seen that the leading rep  $[\lambda]$  is  $[4^5]$  in both cases: in  $s^4 p^{11}(\text{sd})^5$  it arises from the out-product  $[4]o[4^2 3]o[41]$  and in  $s^4 p^{12}(\text{sd})^3(\text{pf})^1$  from



$[4] \circ [4^3] \circ [3] \circ [1]$ . We now ask what reps of  $SU_3$  are consistent with these  $[\lambda]$  out-products.

To answer this we first apply the usual subduction to the  $[\lambda]$  representing each shell in the out-products. Taking out-plethysms from Table 2-2, sifted with mesh 3, the results are as follows.

$$[0] \bar{\circ} [4] = [0],$$

$$[1] \bar{\circ} [4^2_3] = [4^2_3],$$

$$[1] \bar{\circ} [4^3] = [4^3],$$

$$[2] \bar{\circ} [41] = [91] + [82] + [73] + [721] + [64] + [631] + 2[62^2] + [541] + [532] + [4^2_2],$$

$$[2] \bar{\circ} [3] = [6] + [42] + [2^3],$$

$$[3] \bar{\circ} [1] = [3].$$

We then take out-products of the YD's  $[\rho]$  representing each shell. But we have already established that we can neglect inactive shells in so doing. Moreover we will be interested only in the leading YD's  $[\rho]$  in each out-product, and it is not difficult to see that these are obtained from the leading terms in each of the factors of the out-products. Thus we need consider only the following.

$$s^4 p^{11}(\text{sd})^5: \quad [4^2_3] \circ [91] = [13\ 5\ 3] + [13\ 4^2] + [12\ 5\ 4];$$

$$s^4 p^{12}(\text{sd})^5(\text{pf})^1: \quad [6] \circ [3] = [9] + [81] + [72] + [63].$$

The leading reps of  $SU_3$  are therefore  $[13\ 5\ 3] = (82)$  for  $s^4 p^{11}(\text{sd})^5$ , and  $[9] = (90)$  for  $s^4 p^{12}(\text{sd})^5(\text{pf})^1$ ; (90) also occurs in  $s^4 p^{11}(\text{sd})^5$ . The negative-parity bands of the experimental spectrum are consistent with these identifications. For (90),  $K = 0$  only, and  $L = 1, 3, 5, 7, 9$ ; for (82) there should be two bands:  $K = 2$  with  $L = 2, 3, 4, \dots, 10$ , and  $K = 0$  with  $L = 0, 1, \dots, 8$ . As in the case of the  $K = 2$  band of (80) already mentioned, the  $K = 0$  band of (82) is not observed, and the model provides no reason why its energy should be higher.



The leading state of the excited configurations is therefore labelled  $[\lambda] = [4^5]$ ,  $(\lambda\mu) = (82)$ . We will verify explicitly in the next chapter that this state is not spurious, in a sense to be described there. We now consider the question of how high in the spectrum this first excited-configuration state should be found. First we note that it should be one oscillator quantum higher than the ground state. The value of  $\hbar\omega$  is in some treatments taken as an adjustable parameter; but here we are trying to understand the workings of the model rather than trying to obtain a good fit, hence we prefer to keep only a single adjustable parameter  $\mathcal{Q}$ . We therefore adopt a general estimate due to Moszkowski (Mz57):  $\hbar\omega = 41 A^{-1/3} \text{ Mev} = 15.1 \text{ Mev}$  for  $A = 20$ . On this basis the harmonic oscillator shell model without any residual interaction would predict an excitation energy of 15.1 Mev above ground for the lowest abnormal-parity state in  $\text{Ne}^{20}$ . Experimentally this state is found at 4.97 Mev, i.e. 10.1 Mev lower than expected. We have shown in Section 4.2 that the supermultiplet model does not explain any such lowering, since this state and the ground state both belong to  $[\lambda] = [4^5]$  (Table 3-2).

We now point out that the  $\text{SU}_3$  model does provide a mechanism for such a lowering of the leading excited-configuration state compared to the ground state, although there is no quantitative agreement in this example. Referring to Table 5-1 we see that the leading excited-configuration state has  $B = 219$ , hence is favoured by the  $\text{SU}_3$  model compared to the ground state with  $B = 176$ . The difference is

$$\frac{\hbar^2 \Delta B}{3\mathcal{Q}} = \frac{43}{3 \times 3.6} \text{ Mev} = 4.0 \text{ Mev},$$

if we use the overall average value of  $\mathcal{Q}$  obtained above. This effect therefore explains 40% of the 10.1 Mev observed lowering.

Actually these order of magnitude agreements are more significant than they seem: to fit with a single adjustable parameter all fifteen observed





states involving five bands and two or three configurations is a stringent test for any model.

A calculation for the (80) band of  $\text{Ne}^{20}$  involving a closely related model has been published by Talman (Ta62). An older calculation for this same band had been used as an illustration in Elliott's first  $\text{SU}_3$  paper (E58). The alpha-particle model has also been applied to these data by Bouten (Bo62).

Harvey and Elliott (Hv63, E62) have tabulated the reps  $(\lambda\mu)$  predicted for the lowest excited-configuration state for even  $A$  from 16 to 40, without indicating how they are obtained. The technique described in this section does confirm all these  $(\lambda\mu)$  assignments. Harvey has reported (Hv63a) but not yet published some detailed calculations involving the negative-parity states in  $\text{O}^{16}$ ,  $\text{O}^{17}$ ,  $\text{O}^{18}$ , and  $\text{F}^{19}$ ; in this work the  $\text{SU}_3$  model is used to select states which are then used as zero-order states for the usual type of shell-model perturbation calculation, with some success.

We now summarize the important points of this section. We have described a technique for extending the  $\text{SU}_3$  classification to excited configurations, and applied the  $\text{SU}_3$  model both to the ground and to the excited configurations of  $\text{Ne}^{20}$ , comparing predictions in some detail to the large body of experimental data available. We have stressed those predictions of the  $\text{SU}_3$  model which go beyond the rotational model (prediction of the relative positions of the bands) and found the same sort of semi-quantitative agreement there as in the intra-band spacings where the  $\text{SU}_3$  model merely repeats rotational model predictions. Finally we have pointed out that in contrast to the supermultiplet model, the  $\text{SU}_3$  model can explain some lowering of excited-configuration states in the spectrum below their pure harmonic oscillator positions, and in fact accounts for 40% of the observed lowering in  $\text{Ne}^{20}$ , if we use only one adjustable parameter  $\mathcal{Q}$  in the entire analysis.





#### 5.4 On the Uses and Limitations of $SU_3$ Theory

We have shown in the previous section that the  $SU_3$  model gives a rather good qualitative account of the extensive experimental data available on the spectrum of  $Ne^{20}$ . Such a treatment is however an entirely unusual application of the model. One reason for this is experimental: of the few nuclear spectra studied with such thoroughness, fewer still, in the light nuclei, have shown such a wealth of rotational bands. But the other reasons reveal limitations of the model and we will examine them more closely in this section.

Actually there are two different types of use to which this model can be put, and the limitations to be discussed are not so critical for one type as for the other. One type of use is simply based on the fact that the model has some physical relevance, as a shell model of actual nuclei (not only as a shell model of the rotational model). It therefore can be used as a convenient tool to select or construct states which will sometimes resemble the low-lying states of actual nuclei. These can then serve as zero-order states in a shell-model calculation in which further residual interactions are introduced in an attempt to reproduce experimental data as exactly as possible. Techniques have been and are being developed to take advantage of the special properties of  $SU_3$  eigenfunctions in such calculations (M62, 62b, etc.) but will not be studied here. The fact that important residual interactions (e.g. the spin-orbit interaction mentioned below) are not diagonal in this representation, so that the end results may involve considerable mixing of the  $SU_3$  states and may no longer resemble rotational model results, is of no particular concern. If good agreement with experiment is obtained a meaningful contribution is to be credited to the independent-particle model in general,



and to the  $SU_3$  classification in particular insofar as the mixing of states has not been too great.

The other type of use of the model depends not so much on its physical relevance as on the fact that it reproduces rotational model results, and therefore helps us to understand the connexion between the independent-particle approach and the collective approach. This second type of use is stressed in this thesis. The connexion between models established by Elliott is undoubtedly very interesting but it remains of limited applicability for the following reasons.

The mass region where the rotational model has been found to agree substantially with experimental results are  $A > 222$ ,  $150 < A < 190$ , and to a certain extent  $A \sim 25$ . But the assumptions of Elliott's  $SU_3$  formalism restrict its immediate applicability to a few nuclei in the region  $A \sim 25$ , leaving the other two typically rotational regions unexplained.

One of these limitations is not serious. The model as stated so far assumes that total isospin is a good quantum number. Though charge independence of the specifically nuclear forces is undoubtedly an excellent assumption, the ordinary Coulomb force obviously distinguishes between protons and neutrons: its effect increases with  $A$  and is no longer negligible in higher shells. For a heavy nucleus the independent-particle model must be applied to protons and neutrons separately, with the two species filling different shells. We could still classify say  $k_p$  protons in a shell of space degeneracy  $s$  according to the groups  $SU_s$  and, by subduction,  $SU_3$ ; and similarly for  $k_n$  neutrons in a shell of degeneracy  $s' \neq s$ . In each case however we would have to select from the list of reps  $[\lambda]$  of  $SU_s$  or of  $SU_{s'}$  only those whose associates label reps of the spin-isospin group  $SU_4$  that allow the appropriate particular values of total isospin  $T$ , e.g.  $T = \frac{1}{2}k_p$



or  $\frac{1}{2}k_n$  for a shell containing only protons or only neutrons. The end results for each of the two active shells could then be combined by the out-product operation as in Section 5.3, thus providing an  $SU_3$  classification and hence rotational bands.

It seems hardly worth while however to set up such a formalism in more detail, because a more serious limitation of the  $SU_3$  model comes into play simultaneously. This more serious limitation is the fact that when it becomes unreasonable to neglect Coulomb forces, the assumption of LS coupling must also be rejected because of the importance of the spin-orbit interaction in heavy nuclei, as represented for instance by the single-particle spin-orbit term in the hamiltonian of the Mayer-Jensen model (E57).

Without entering the controversy as to whether jj coupling might not be as appropriate as LS coupling for some light nuclei, we note the obvious fact that for the rotational regions  $A > 222$  and  $150 < A < 190$  the magic numbers of the Mayer-Jensen model, 50, 82, 126, are immensely more relevant than those of the LS-coupling harmonic oscillator model, 40, 70, 112. It may be possible to adjust the form of the hamiltonian so as to obtain the experimentally observed magic numbers without giving up LS coupling, but one then loses the special symmetries of the harmonic oscillator, upon which the Elliott treatment depends. It would clearly be desirable to find a group-theoretical classification that would obtain rotational bands from the independent-particle model in jj coupling, in the presence of spin-orbit interaction (as in the Mayer-Jensen model). This problem is unsolved.

We might also consider as a limitation on the ability of the  $SU_3$  model to predict actual rotational bands the fact that it involves the orbital angular momentum  $L$  instead of the total (orbital plus spin) angular momentum  $J$ . Only in the special case  $S=0$  can the bands of values of  $L$  predicted





by the  $SU_3$  classification be immediately re-interpreted as bands of values of  $J$ , as in the example of  $Ne^{20}$ . This already excludes all odd- $A$  nuclei, for which  $S$  can not be integral, as well as many states of even- $A$  nuclei for which  $S = 1, 2$ , etc.

It is true that even for  $S > 0$  the prediction of a band of values of  $L$  amounts to the prediction of a set of values of  $J$ , which can be divided into subsets satisfying the rules

$$J = 0, 2, 4, 6, \dots \quad \text{if } K_J = 0^+,$$

$$J = 1, 3, 5, 7, \dots \quad \text{if } K_J = 0^-,$$

$$J = K, K+1, K+2, \dots \quad \text{if } K_J > 0,$$

for suitably chosen values of  $K_J$ . This follows simply from the triangle inequalities governing the addition of angular momenta. Thus, writing now  $K_L$  for the quantity previously denoted by  $K$ , one can easily verify that for  $S = \frac{1}{2}$ , an  $L$ -band with  $K_L = 0^+$  or  $0^-$  implies a  $J$ -band with  $K_J = \frac{1}{2}$ , and an  $L$ -band with  $K_L > 0$  implies two  $J$ -bands with  $K_J = K_L \pm \frac{1}{2}$ . Similarly if  $S = 1$ ,  $K_L = 0^+$  implies three bands  $K_J = 0^+, 0^-$ , and  $1$ ;  $K_L = 0^-$  implies two bands  $K_J = 0^+$  and  $1$ ;  $K_L = 1$  implies four bands  $K_J = 0^+, 0^-, 1$ , and  $2$ ; and  $K_L > 1$  implies three bands with  $K_J = K_L$ , and  $K_L \pm 1$ ; and so on for higher  $S$ . (The 'parities' attached to values of  $K_J = 0$  above are meant only to distinguish bands containing odd  $J$  values only from those containing even  $J$  values only, and should not be taken seriously as parities.)

However, even if 'J-bands' are obtained in this way, the  $SU_3$  model as it stands, based on Moszkowski's quadrupole-quadrupole residual interaction, still predicts an energy dependence proportional to  $L(L+1)$ , not  $J(J+1)$ . There does not seem to be any experimental indication of bands actually having  $L(L+1)$  energy spacing rather than  $J(J+1)$ , for  $S > 0$ . At the end of his recent Mexican lectures (E62), Elliott has begun to examine the



question of whether an additional interaction might change the  $L(L+1)$  to the desired  $J(J+1)$  dependence. These preliminary considerations indicate the desired effect might be obtained through the 'delicate' assumption of a mutual cancellation of two unwanted quantities. While such a notion may be useful in an attempt to fit a particular spectrum (first type of use of the model), it is not very helpful in obtaining a general understanding of how the shell model can be made to predict the rotational properties clearly exhibited by many nuclei (second type of use). If and when a solution is obtained to the problem of finding a  $jj$ -coupling classification with rotational properties in the presence of spin-orbit interaction, the  $J(J+1)$  dependence should appear in a more natural manner.

Another limitation involves the 'parities' that have to be assigned to the  $K = 0$  bands (both for  $K_L$  and for  $K_J$  as above) to account for the fact that either even values of  $J$  only or odd values of  $J$  only appear in these bands. The idea is that in the simple rotational model the quadrupole type of distortion from sphericity will produce only even  $J$  for  $K = 0$ , and the rotational model eigenfunctions for such distortions have positive parity, whereas only octupole (or higher) distortion can produce  $K = 0$  bands with odd  $J$  only, and these eigenfunctions have negative parity. But the parity of the  $SU_3$  eigenfunctions on the other hand is simply that of the  $r$ -quanta harmonic oscillator eigenfunctions, well known to be  $(-1)^r$ , and this does not always agree with the 'parities' described above. For example in the system  $A = 10$ , ground configuration  $p^6$ , all  $SU_3$  eigenfunctions have positive parity  $(-1)^6$ , but both  $K_L = 0^+$  and  $K_L = 0^-$  bands are predicted:  $K_L = 0^+$  in the rep  $[\lambda] = [42]$ ,  $(\lambda\mu) = (22)$ , and  $K_L = 0^-$  in the rep  $[\lambda] = [41^2]$ ,  $(\lambda\mu) = (30)$ .

Having mentioned various differences between the detailed predictions



of the  $SU_3$  model and those of the rotational model, we repeat here for completeness one very obvious difference. The predicted bands of the  $SU_3$  model are cut off at some upper value of  $L$ , whereas those of the rotational model are not. This may seem at first sight an opportunity for a crucial experimental test of the  $SU_3$  model, but actually it is not so simple to test. Experimentally observed rotational bands seldom contain more than about four states and the situation naturally becomes more confused with increasing excitation energy, as more bands, more transitions, and more configurations come into the picture.

It may be appropriate to end this section with some remarks comparing the  $SU_3$  model with the Nilsson model. The Nilsson model provides a set of single-particle eigenfunctions resulting from a central field already embodying a departure from spherical symmetry. The most direct application of this is to treat a nucleus as a single nucleon around a core of  $A-1$  nucleons. The Nilsson orbits then provide intrinsic states, i.e. values of  $K_J$ , at specified excitation energies, each being the lowest state of a rotational band. Naturally this type of description is most immediately applicable to odd- $A$  nuclei (e.g. see Braben et al. (Bb62) on  $Na^{23}$ ), for which the corresponding predictions of the  $SU_3$  model involve the uncertain passage from  $L$ -bands to  $J$ -bands discussed above.

More sophisticated things can also be done with the Nilsson model, abandoning the picture of a single nucleon outside a core of  $A-1$  nucleons. One particular type of application will be mentioned because of its close connexion with the straightforward  $SU_3$  model. If one considers many-particle functions formed from products of Nilsson single-particle functions, and then handles them in the same way as Elliott (E58) handles his many-particle functions formed from products of harmonic oscillator functions,





projecting from them states of definite  $J$ , the states obtained are sometimes found to be useful. This method has been applied by Kurath and Picman (Ku59) in the  $p$  shell, by Redlich (Re58) in the  $(sd)$  shell (and in particular by Talman (Ta62) for  $\text{Ne}^{20}$ ) and by Lawson (La61, 62) and Zeidman (La62) in the  $f 7/2$  shell.

In summary, the  $SU_3$  model and techniques deriving from it have been found to be a useful specialized tool in several particular applications of the independent-particle model. When the  $SU_3$  model is viewed as a means of understanding how the independent-particle model can be made to yield rotational model results, its applicability is severely restricted by the LS-coupling assumption, which prevents its application to heavy nuclei.





## Chapter 6      EQUIVALENT APPROACHES

### 6.1      Kretzschmar's Treatment and Spurious States

The  $SU_3$  classification has been presented so far in the formulation originally given by Elliott, i.e. as a subclassification of the supermultiplet classification applied to the  $k$  particles of the active shell in ground configurations. We have already seen in this thesis how this formalism is easily extended to apply to the  $A$ -particle functions for both ground and excited configurations. Several treatments have been given, either independent of or subsequent to Elliott's, which obtain either the same or equivalent classifications in different ways. In this chapter we review two such approaches which we have found particularly interesting.

We consider in this first section the work of M. Kretzschmar (K60, 60b), commenting first on the way he sets up the classification procedure, and then applying to the case of  $Ne^{20}$  the technique which he develops for handling the problem of spurious states.

Kretzschmar does not claim independent rediscovery of Elliott's classification. His papers seem to have two purposes: to emphasize and generalize the mathematical techniques of the classification procedure, showing how it can be set up in different ways, and to give a group-theoretical solution to the problem of spurious states. His efforts towards the first goal have been useful because the techniques are unknown to most physicists and have been markedly de-emphasized by Elliott, and even more so by other workers in this field; and also because they make it easier to see the relationships between independent derivations of the  $SU_3$  classification, as



we illustrate in the next section. Much of the present thesis can be considered as a continuation of these efforts.

Kretzschmar introduces the supermultiplet classification only as a final step, after the  $SU_3$  and angular momentum classifications. Instead of starting, as Elliott did, from the degeneracy of the single-particle hamiltonian, expressed by the quantity  $s$  which depends on the particular active shell involved, Kretzschmar starts from the degeneracy of the  $A$ -particle hamiltonian. He does not explicitly introduce the shell structure, which of course is implicit in his choice of the harmonic oscillator hamiltonian, and which reveals itself 'automatically' at the end of the treatment in a formally interesting way; thus the entire development involves all  $A$  particles rather than the  $k$  particles of the active shell, and ground and excited configurations are on the same footing.

We have emphasized in Section 5.1 that the harmonic oscillator eigenfunctions are homogeneous polynomials of degree  $r$  in three creation operators  $a_x^*$ ,  $a_y^*$ ,  $a_z^*$ . The group  $SU_3$  is then introduced as the group of unitary transformations in the complex three-dimensional linear vector space of which the three  $a_i^*$  are unit vectors. Kretzschmar proceeds in this way except that he gives the creation operators a particle label as well as the  $xyz$  label. For  $A$  nucleons he thus has homogeneous polynomials of degree  $r$  in  $3A$  creation operators, hence a complex  $3A$ -dimensional linear vector space; the symmetry group of his  $A$ -nucleon hamiltonian is  $SU_{3A}$ . Because all  $3A$  creation operators commute, the wave functions must belong to the rep of  $SU_{3A}$  denoted by the single-row YD  $[r]$ , i.e. the completely symmetric rep of the permutation group  $S_r$ .

The next step is to consider a subgroup of  $SU_{3A}$ , namely the direct product  $SU_3 \times SU_A$ . The subduction  $SU_{3A} \rightarrow SU_3 \times SU_A$  will break



down the rep  $[r]$  of  $SU_{3A}$  into reps of  $SU_3 \times SU_A$ . But any rep of  $SU_3 \times SU_A$  is necessarily given by the outer Kronecker product of a rep of  $SU_3$ , say  $[\rho]$ , by a rep of  $SU_A$ , say  $[\rho']$ , where  $\#[\rho] = \#[\rho'] = r$ .

In the dual interpretation,  $[r]$ ,  $[\rho]$ , and  $[\rho']$  are all reps of the same group  $S_r$  of 'permutations of the quanta'. The YD operation involved is therefore the in-product, which we have introduced in Chapter 2 as corresponding to the inner Kronecker product of two reps of the same symmetric group (here  $S_r$ ). We are interested in finding out in what circumstances the special rep  $[r]$  occurs in the in-product  $[\rho] \cdot [\rho']$ . The answer is implied by a special property of the in-product operation that we mentioned in Section 2.2: the in-product  $[\rho] \cdot [\rho']$  contains  $[r]$  if and only if  $[\rho] = [\rho']$ . Conversely the rep  $[r]$  of  $SU_{3A}$  breaks down into a direct sum of reps of  $SU_3 \times SU_A$  formed by taking in-products of a rep of  $SU_3$  labelled by  $[\rho]$  with a rep of  $SU_A$  labelled by the same YD  $[\rho]$ . It follows that the only acceptable YD's  $[\rho]$  satisfy both  $NR[\rho] \leq A$  and  $NR[\rho] \leq 3$ . For  $A \geq 3$  therefore, this classification is identical to Elliott's classification into reps of  $SU_3$  labelled by  $[\rho]$  (or at least to that version of it which involves  $A$  rather than  $k$  nucleons).

Kretzschmar then introduces the angular momentum classification by a subduction equivalent to  $SU_3 \rightarrow SU_2$  (though neither he nor Elliott takes advantage of the out-plethysm technique for this subduction as we have done in Section 3.3).

In his formalism the introduction of the supermultiplet classification is the final step. For this purpose he considers  $[\rho]$  as a rep of  $SU_A$  rather than  $SU_3$ , and introduces a subgroup  $H$  of  $SU_A$  consisting of the permutation matrices (with a single matrix element in each row or column equal to one, and zeros elsewhere).  $H$  is therefore isomorphic to the





permutation group  $S_A$ , and the subduction  $SU_A \rightarrow H$  or  $SU_A \rightarrow S_A$ , leading to the supermultiplet classification in terms of YD's  $[\lambda]$  of  $A$  boxes, is then obtained by straightforward application of the in-plethysm technique described in Section 2.4. Thus we end up with states labelled by  $r$ ,  $[\rho]$ ,  $L$ , and  $[\lambda]$ , just as in the Elliott classification, even though the subclassifications were performed in a different order.

We now come to the matter of spurious states. Throughout the development outlined above, Kretzschmar has carried out a parallel derivation which applies to a slightly different hamiltonian. The usual independent-particle model hamiltonian, such as the harmonic oscillator hamiltonian we have been discussing, relates the motions of nucleons to a coordinate system supposedly fixed in space independently of the positions of the nucleons. But if this hamiltonian is to represent the combined effect on one nucleon of the forces due to all the others, the forces should be made to depend on the position of each nucleon with respect to all the others (or, say, with respect to the centre of mass of the whole nucleus), not with respect to an arbitrary position fixed in space. Elliott and Skyrme (E55) have pointed out that this inconsistency leads to the appearance, in the usual treatment, of spurious states, which involve say a simultaneous motion of the whole nucleus with respect to the fictitious point fixed in space. Such states are of no physical interest. In the case of a harmonic oscillator potential it is possible to identify and eliminate them.

Kretzschmar has shown that the group-theoretical classification procedure described above can be modified in such a way that the spurious states do not arise. By a coordinate transformation it is possible to separate the usual hamiltonian into a part which is invariant under space translations (eliminating the fictitious reference to a point fixed in space) and



another part which is rejected as unphysical. The net effect of using the translation-invariant hamiltonian instead of the usual one is roughly to replace  $A$  by  $A-1$  throughout the classification procedure. (We lose 3 of the  $3A$  degrees of freedom by rejecting as meaningless the motions of the centre of mass of the nucleus with respect to the fictitious point fixed in space.)

In particular it turns out that the last step of the classification procedure, which involved the subduction  $SU_A \rightarrow S_A$ , now involves  $SU_{A-1} \rightarrow S_A$  instead. This subduction is accomplished by calculating the in-plethysm  $[A-1, 1] \tau [\rho]$ , which therefore leads only to non-spurious states  $[\lambda]$ . (The usual formulation leads to  $\{[A] + [A-1, 1]\} \tau [\rho]$ , which produces both spurious and non-spurious states  $[\lambda]$ .)

In order to check whether a given state labelled by  $[\lambda]$  and  $[\rho]$  is spurious therefore, we can check whether that particular YD  $[\lambda]$  occurs in the in-plethysm  $[A-1, 1] \tau [\rho]$ , or  $[c1] \tau [\rho]$ . If it does we know that the state is physically significant and not spurious.

We now apply this check to the state  $[\lambda] = [4^5]$ ,  $[\rho] = [1353]$ , i.e.  $(\lambda\mu) = (82)$ , which we claimed in Section 5.3 to be the lowest negative-parity state in  $\text{Ne}^{20}$ . The purpose of checking this particular example is the following. We have seen that the position of the lowest negative-parity state in the spectrum is not affected by the supermultiplet model, and is lowered by the  $SU_3$  model. It is not affected by the supermultiplet model because this state belongs to the same rep  $[\lambda]$  as the ground state, hence has the same value of  $\Xi$ . Now in general ground-configuration states are never spurious. If  $[\lambda] = [4^5]$ ,  $(\lambda\mu) = (82)$  is spurious, the lowest negative-parity state would have to belong to another  $[\lambda]$ , with a  $\Xi$  different from that of the ground state. Hence the supermultiplet model



would affect its position with respect to the ground state and spoil the line of argument developed in Sections 4.2 and 5.3.

We therefore want to find out whether  $[\lambda] = [4^5]$  occurs in the in-plethysm  $[c1] \cdot [1353]$ . A complete evaluation of this in-plethysm would be tedious but we can answer this particular question without carrying out the whole procedure. (The reader who wishes to follow the details of this verification should perhaps review the relevant part of Section 2.4.)

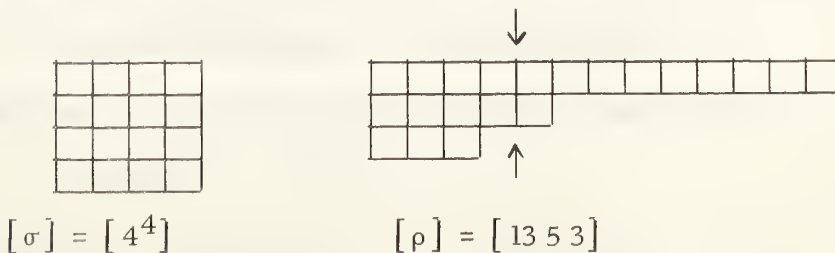
We have seen in Section 2.4 that the YD  $[\lambda] = [c\sigma]$  occurs in the in-plethysm  $[c1] \cdot [\rho]$  if and only if  $[\sigma]$  occurs in the expression obtained by applying the operator  $\mathcal{D}$  to  $[\rho]$ . In this example therefore  $[\lambda] = [4^5] = [c4^4]$  occurs in  $[c1] \cdot [1353]$  if and only if  $[\sigma] = [4^4]$  occurs in  $\mathcal{D}[1353]$ .

Now we have given an expansion of  $\mathcal{D}$  into parts  $\Delta_\ell$  such that each part, applied to  $[\rho]$ , produces only YD's  $[\sigma]$  having a certain number of boxes:

$$\#[\sigma] = \#[\rho] - \ell.$$

In the present example  $\#[\sigma] = \#[4^4] = 16$ , and  $\#[\rho] = \#[1353] = 21$ . The term we are looking for can therefore be produced only by  $\Delta_5$ . (We neglect temporarily the matter of modification rules.)

At this point it is convenient to look at pictures of  $[\sigma]$  and  $[\rho]$ .



Now each of the many terms in  $\Delta_5$  contains one or more Foulkes operators, which remove boxes from  $[\rho]$ , and in general one or more YD's by which the result is then to be out-multiplied, restoring (perhaps in different places) all but five of the boxes removed from  $[\rho]$  by Foulkes operators. In





particular if we are to obtain  $[\sigma] = [4^4]$  from  $\Delta_5[\rho] = \Delta_5[13\ 5\ 3]$ , this must be accomplished by a term of  $\Delta_5$  containing Foulkes operators capable of removing the ten boxes appearing to the right of the arrows in the picture of  $[\rho]$ , since these do not appear in  $[4^4]$ .

Now a Foulkes operator  $D([\mu])$  removes  $\# [\mu]$  boxes. Hence we need a Foulkes operator in one of the terms of  $\Delta_5$ , whose argument has at least ten boxes (or perhaps a combination of Foulkes operators in the same term of  $\Delta_5$  whose arguments total at least ten boxes.) Now from the explicit expressions 2.47 given for  $\Delta_5$  in Section 2.4, it is easily seen that only the last few terms of  $\Delta_5$  satisfy this condition, namely

$$\begin{aligned} \Delta_5 = & \dots + [5] \circ D([2] \overline{o} [5]) + [41] \circ D([2] \overline{o} [41]) + [32] \circ D([2] \overline{o} [32]) \\ & + [31^2] \circ D([2] \overline{o} [31^2]) + [2^21] \circ D([2] \overline{o} [2^21]) + [21^3] \circ D([2] \overline{o} [21^3]) \\ & + [1^5] \circ D([2] \overline{o} [1^5]). \end{aligned}$$

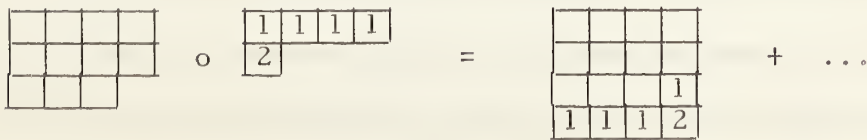
Any one of these Foulkes operators can remove ten boxes from  $[\rho]$ .

Now to remove nine boxes in the first row and one in the second, it is not difficult to see that we need the Foulkes operator  $D([91])$ . Because  $[4^23] \circ [91] = [13\ 5\ 3] + \dots$ , it follows that  $D([91])[13\ 5\ 3] = [4^23] + \dots$ . From Table 2-2 and the expression for  $\Delta_5$  above, we see that the Foulkes operator  $D([91])$  occurs in the term

$$[41] \circ D([2] \overline{o} [41]) = [41] \circ D([91]) + \dots$$

When this is applied to  $[\rho] = [13\ 5\ 3]$ , we get a term

$$[41] \circ [4^23] = [4^4] + \dots$$



Thus we have shown that a term  $[\sigma] = [4^4]$  appears in the expression  $\mathcal{D}[13\ 5\ 3]$ . From the theorem of Section 2.4, this produces a term





$[c4^4]$  in the in-plethysm  $[c1] \cdot [1353]$ . But  $S_A$  is  $S_{20}$  for  $\text{Ne}^{20}$ , so that  $[c4^4] = [4^5]$ , to make  $[\lambda]$  a YD of 20 boxes. Hence we have almost completed the proof that a term  $[4^5]$  appears in the in-plethysm  $[c1] \cdot [1353]$ .

To complete the proof we must verify that this term  $[4^5]$  is not cancelled by another term  $[4^5]$  produced from a different part of  $\mathcal{D}[1353]$  and bearing a minus sign because of modification rules. Because  $\#[\lambda] = 20$ ,  $\#[\rho] = 21$ , and  $\#[\sigma] = \#[\rho] - \ell = 21 - \ell$ , such a contribution would have to be (before modification) of the form  $[\lambda] = [\ell-1, \sigma_1, \sigma_2, \dots]$ , which gives  $-\ell[\sigma_1 - 1, \ell, \sigma_2, \dots]$  under the action of the modification rules, provided  $\ell < \sigma_1$ . (If  $\ell = \sigma_1$ , the modification rules would make the term vanish; if  $\ell > \sigma_1$ , then  $[\lambda]$  is already in standard order and no modification is necessary, hence no minus sign can appear.) But if  $-\ell[\sigma_1 - 1, \ell, \sigma_2, \dots]$  is to equal  $-[4^5]$ , we must have  $\sigma_1 = 5$ , hence  $\ell < 5$ .

But to obtain a  $[\sigma]$  with  $\sigma_1 = 5$  from  $\mathcal{D}[1353]$  requires that at least eight boxes be removed from the first row by some Foulkes operator, and no Foulkes operator appearing in the explicit expressions for  $\Delta_\ell$ ,  $\ell = 0, 1, 2, 3$ , is capable of this. If  $\ell = 4$ , the required  $[\sigma]$  would have to be  $[54^3]$  in order to produce  $[c\sigma] = [354^3] = -[4^5]$ . But to obtain  $[54^3]$  from  $\Delta_4[1353]$  requires some Foulkes operator to remove at least eight boxes from the first row and one box from the second row of  $[1353]$ . No Foulkes operator or combination of Foulkes operators appearing in  $\Delta_4$  is capable of this.

The conclusion is that no term  $-[4^5]$  can occur to cancel the term  $+ [4^5]$  which we have shown to be produced from  $\Delta_5$ . Hence  $[4^5]$  does occur in the in-plethysm  $[c1] \cdot [1353]$ . This means that the state  $[\lambda] = [4^5]$ ,  $(\lambda\mu) = (82)$  is not spurious, and is indeed the leading excited-configuration state in  $\text{Ne}^{20}$ . Because it has the same  $[\lambda]$  as the ground state



the supermultiplet model does not affect its position in the spectrum in any way, and the application of the  $SU_3$  model to the position of this state suffers no interference from the supermultiplet model, confirming the claim we made in Section 5.3. The verification would of course have been trivial if a tabulation had existed, extending Table 2-4 to sufficiently large YD's. The necessary calculations would then have been done once and for all and could be used in other eventual physical applications of in-plethysm.

## 6.2 The Pseudo-Spin Approach: $SU_A$ , $SU_3$ , and $SU_2$

Another interesting formulation of the  $SU_3$  classification has been given by Bargmann and Moshinsky (B60, 61). These authors were led to the problem by Moshinsky's earlier investigations (M59, 60, Bd60, 60a) on the subject of 'transformation brackets'. In this section we present and comment upon the results of a calculation carried out independently in the formalism set up by Moshinsky. We then use the ideas developed by Kretzschmar and described in Section 6.1 to show the precise relationship between the  $SU_A$  classification of Bargmann and Moshinsky and the usual  $SU_3$  classification.

Consider two particles in a three-dimensional isotropic harmonic oscillator potential, with orbital angular momenta  $\ell_1$  and  $\ell_2$ , total orbital angular momentum  $L$  (instead of  $\lambda$  in Moshinsky's papers), and radial quantum numbers  $n_1$  and  $n_2$ . The radial quantum number  $n$  used here is smaller by one unit than the more usual one: the first occurrence of each value of  $\ell$  in the spectrum is labelled  $n=0$ , not  $n=1$ . The number of quanta  $N$  for each level is therefore  $N = 2n + \ell$ , and the total energy quantum



number  $r$  (which Moshinsky writes  $\rho$ ) is

$$r = 2n_1 + l_1 + 2n_2 + l_2. \quad \underline{6.1}$$

The possible single-particle states  $n, l$  for the first few levels  $N$  are listed below.

$N = 5$	$n, l =$	2, 1	1, 3	0, 5
4		2, 0	1, 2	0, 4
3		1, 1	0, 3	
2		1, 0	0, 2	
1		0, 1		
0		0, 0		

Moshinsky is interested in a coordinate transformation which enables him to switch back and forth between systems whose origins are the centre of the harmonic oscillator well, and the centre of mass of the two nucleons, respectively. He has noticed that the form of the harmonic oscillator hamiltonian, and hence the form of his single-particle eigenfunctions, is invariant under this transformation and he is interested in the behaviour of the two-particle eigenfunctions having total energy quantum number  $r$  and total orbital angular momentum  $L$  under this transformation. The coefficients expressing this behaviour form matrices  $\Delta^{rL}$ , whose elements are the transformation brackets, or facetiously the brashinskets.

Now the two-particle eigenfunctions are homogeneous polynomials in  $3A = 6$  creation operators. Whereas Elliott ignored the nucleon labels and considered the eigenfunctions as homogeneous polynomials in three variables  $a_x^*, a_y^*, a_z^*$ , Moshinsky does exactly the opposite: he ignores the  $x y z$  label on the six creation operators and considers the eigenfunctions as homogeneous polynomials in two variables. Viewed in this way his





coordinate transformation is an element of the group  $SU_2$ , acting on the linear vector space whose unit vectors are the two creation operators  $a_1^*$  and  $a_2^*$  (which stand for any of  $a_{x1}^*$ ,  $a_{y1}^*$ ,  $a_{z1}^*$ , and any of  $a_{x2}^*$ ,  $a_{y2}^*$ ,  $a_{z2}^*$  respectively). He has pointed out that the hamiltonian is invariant under all elements of this group  $SU_2$ , and hence that his matrices  $\Delta^{rL}$  can be generalized to form representations of  $SU_2$ , (in general reducible). He also derives the dimension  $\tau(rL)$  of the representation  $\Delta^{rL}$ , which is simply the number of two-particle states labelled  $n_1, l_1, n_2, l_2$ , consistent with given values of  $r, L$ .

We now obtain Moshinsky's formula for  $\tau(rL)$  in a slightly different way, and then give the reduction of  $\Delta^{rL}$  into a direct sum of reps of  $SU_2$ .

For fixed  $r$  and  $L$ , the possible values of  $n_1, l_1, n_2, l_2$  are restricted by

$$2n_+ + l_+ = 2n_1 + l_1 + 2n_2 + l_2 = r, \quad 6.2$$

$$\text{and} \quad |l_1 - l_2| \leq L \leq l_+, \quad 6.3$$

$$\text{where} \quad l_+ = l_1 + l_2, \quad \text{and} \quad n_+ = n_1 + n_2. \quad 6.4$$

Because  $l_1 \geq L$ , we have  $2n_+ = r - l_+ \leq r - L$ ; or  $n_+ \leq \frac{1}{2}(r - L)$ .

Moreover for fixed  $n_+$  we have  $n_1 = 0, 1, \dots, n$ , and  $n_2 = n_+ - n_1$  is then determined.

Finally, for fixed  $n_+$  and  $n_1$ ,  $l_+ = r - 2n_+$  is also fixed. The numbers  $l_1$  and  $l_2$  obey  $l_1 + l_2 = l_+$ , so that we vary only  $l_1$  and consider  $l_2 = l_+ - l_1$  determined. Consider the restriction  $|l_1 - l_2| \leq L$ , or  $|l_+ - 2l_1| \leq L$ . For the lowest  $l_1$ , we must have  $l_1 \leq \frac{1}{2}l_+$ ; otherwise the corresponding  $l_2 = l_+ - l_1 \leq l_+ - \frac{1}{2}l_+ = \frac{1}{2}l_+$ , and by symmetry between the two nucleons there would have to be a state with  $l_1$  equal to this value, thus contradicting the assumption that we had chosen the lowest  $l_1$ . Therefore the restriction reads  $l_+ - 2l_1 \leq L$ , and  $l_1 \geq \frac{1}{2}(l_+ - L) = \frac{1}{2}(r - 2n_+ - L)$ . Similarly, the



highest value of  $\ell_2$  must be  $\geq \frac{1}{2}\ell_+$ . Hence  $2\ell_1 - \ell_+ \leq L$ .

To summarize, all states consistent with given  $r, L$  are enumerated if we let  $\ell_1, n_1$ , and  $n_+$  vary in integral steps over the following ranges of values.

If  $r-L$  is even:

$$\frac{1}{2}(\ell_+ - L) \leq \ell_1 \leq \frac{1}{2}(\ell_+ + L),$$

$$0 \leq n_1 \leq n_+,$$

$$0 \leq n_+ \leq \frac{1}{2}(r-L);$$

and if  $r-L$  is odd:

$$\frac{1}{2}\{\ell_+ - (L-1)\} \leq \ell_1 \leq \frac{1}{2}\{\ell_+ + (L-1)\},$$

$$0 \leq n_1 \leq n_+,$$

$$0 \leq n_+ \leq \frac{1}{2}\{r - (L+1)\}.$$

A triple sum subject to these limits will be denoted by  $\sum_m$ . The calculation of  $\tau(rL)$  is now straightforward:

$$\tau(rL) = \sum_m 1 = \begin{cases} (1/8)(L+1)(r-L+2)(r-L+4) & \text{if } r-L \text{ is even,} \\ (1/8)L(r-L+1)(r-L+3) & \text{if } r-L \text{ is odd.} \end{cases} \quad \underline{6.5}$$

We now consider the reduction of the representation  $\Delta^{rL}$  of  $SU_2$  into a direct sum over the well known reps  $D_f$  of  $SU_2$ , with  $f = 0, \frac{1}{2}, 1, 3/2, \dots$ :

$$\Delta^{rL} = \sum_f M_f^{rL} D_f. \quad \underline{6.6}$$

The coefficients  $M_f^{rL}$  are non-negative integers. The quantity  $f$  is Moshinsky's pseudo-spin, so called because it has the mathematical properties of a spin, but has nothing to do with rotations in the usual physical space; it has nothing to do with the intrinsic spin of the nucleons either.

To obtain the coefficients  $M_f^{rL}$ , we parametrize the group  $SU_2$  as in Wigner's textbook (W59), noting that all elements are equivalent to (i.e. in the same class as) an element of the form

$$U(\alpha) = \begin{bmatrix} e^{-\frac{1}{2}i\alpha} & 0 \\ 0 & e^{\frac{1}{2}i\alpha} \end{bmatrix}. \quad \underline{6.7}$$



All classes are obtained when  $\alpha$  varies continuously over the interval

$$0 \leq \alpha \leq 2\pi.$$

The character of the rep  $D_f$  for the class  $\alpha$  is known to be

$$\text{tr } D_f(\alpha) = \frac{\sin(2f+1)\frac{1}{2}\alpha}{\sin\frac{1}{2}\alpha}. \quad \underline{6.8}$$

Because of the diagonal form of  $U(\alpha)$ , we can easily obtain the corresponding character for the representation  $\Delta^{rL}$ . Since  $\Delta^{rL}$  represents the transformation properties of two-particle eigenfunctions labelled  $n_1, \ell_1, n_2, \ell_2$ , which are of degree  $2n_2 + \ell_2$  in  $a_2^*$  and of degree  $2n_1 + \ell_1$  in  $a_1^*$ , the diagonal elements of  $\Delta^{rL}$  are of the form

$$e^{\frac{1}{2}i\alpha(2n_1 + \ell_1 - 2n_2 - \ell_2)} = e^{i\alpha(\frac{1}{2}r - 2n_2 - \ell_2)}.$$

And the character is

$$\text{tr } \Delta^{rL}(\alpha) = \sum_{\mathbf{m}} e^{i\alpha(\frac{1}{2}r - 2n_2 - \ell_2)} = \sum_{\mathbf{m}} e^{i\alpha(\frac{1}{2}r - 2n_1 - \ell_1)}$$

Substituting into 6.6 for  $\alpha = 0$ , (the identity class), we get

$$\tau(rL) = \text{tr } \Delta^{rL}(0) = \sum_f M_f^{rL} \text{tr } D_f(0) = \sum_f M_f^{rL} (2f+1),$$

$$\text{i.e. } \tau(rL) = M_0^{rL} + 2M_{\frac{1}{2}}^{rL} + 3M_1^{rL} + 4M_{\frac{3}{2}}^{rL} + 5M_2^{rL} + \dots \quad \underline{6.9}$$

Similarly for the class  $\alpha = 2\pi$ , which is distinct from the identity class, the characters are  $\text{tr } \Delta^{rL}(2\pi) = (-1)^r \tau(rL)$ , and  $\text{tr } D_f(2\pi) = (-1)^{2f} (2f+1)$ .

$$\text{Hence } (-1)^r \tau(rL) = \sum_f M_f^{rL} (-1)^{2f} (2f+1),$$

$$\text{or } (-1)^r \tau(rL) = M_0^{rL} - 2M_{\frac{1}{2}}^{rL} + 3M_1^{rL} - 4M_{\frac{3}{2}}^{rL} + 5M_2^{rL} - \dots \quad \underline{6.10}$$

Comparing these two results we see that for even (odd)  $r$ , only integral (half-integral) values of  $f$  will occur in the reduction of  $\Delta^{rL}$ .

Now from 6.6 we obtain

$$M_f^{rL} = \frac{1}{2\pi} \int_0^{2\pi} \text{tr } \Delta^{rL}(\alpha) \text{tr } D_f(\alpha) (1 - \cos\alpha) d\alpha. \quad \underline{6.11}$$



(The meaning of the group integration, with weighting factor  $(1-\cos\alpha)$ , is explained in Wigner's book (W59).) This leads to

$$M_f^{rL} = \frac{1}{4\pi} \sum_m \int_0^{2\pi} e^{i\alpha(\frac{1}{2}r - 2n_1 - \ell_1)} \{ e^{if\alpha} + e^{-if\alpha} - e^{i(f+1)\alpha} - e^{-i(f+1)\alpha} \} d\alpha \quad 6.12$$

Because only integral (half-integral)  $f$  occur with even (odd)  $r$ , we need consider only terms of the form  $\int_0^{2\pi} e^{ib\alpha} d\alpha$ , where  $b$  is an integer, in which case  $\int_0^{2\pi} e^{ib\alpha} d\alpha = 2\pi \delta_{b,0}$ . If we define

$$\sigma = \frac{1}{2}r - 2n_1 - \ell_1 = \frac{1}{2}\{ (2n_2 + \ell_2) - (2n_1 + \ell_1) \}, \quad 6.13$$

we obtain 
$$M_f^{rL} = \frac{1}{2} \sum_m \{ \delta_{\sigma,-f} + \delta_{\sigma,f} - \delta_{\sigma,-(f+1)} - \delta_{\sigma,(f+1)} \}. \quad 6.14$$

Moreover by symmetry between the two nucleons we know that for every  $\sigma > 0$ , the value  $-\sigma$  must also occur with the same frequency in the list of possible two-particle states. And for  $\sigma = 0$ , though the state with  $-\sigma$  is not listed separately, the same state scores simultaneously on two of the Kronecker deltas. Hence

$$M_f^{rL} = \sum_m \{ \delta_{\sigma,f} - \delta_{\sigma,f+1} \}. \quad 6.15$$

In words,  $M_f^{rL}$  is obtained by counting the number of times  $f$  occurs minus the number of times  $f+1$  occurs, in the list of values of  $\sigma$  for all the states  $n_1, \ell_1, n_2, \ell_2$  compatible with given  $r, L$ . But this is exactly analogous to an elementary procedure for obtaining the possible values of an angular momentum quantum number, given a list of possible values of its projection quantum number:  $\sigma$  is acting here as a projection quantum number for the pseudo-spin  $f$ . For each value of  $f$ ,  $\sigma$  ranges in integer steps from  $-f$  to  $+f$ .

Since  $2\sigma\hbar\omega$  is by definition the difference between the energies attributed to the two particles, this provides a physical interpretation of the pseudo-spin. The relevant properties of the single-particle spectrum





are that (a) the levels are evenly spaced in energy, and (b) the number of states  $n, \ell$  for given energy  $N = 2n + \ell$  is a non-decreasing function of  $N$ . From these properties it follows that for every value of  $\sigma$  in the list of possible two-particle states of given  $r, L$ , say  $\sigma'$ , all the values  $\sigma = \sigma' - 1, \sigma' - 2, \dots, -\sigma'$  must also occur: thus the values of  $\sigma$  behave like the eigenvalues of an angular momentum component.

The evaluation of the triple sum  $M_f^{rL} = \sum_m \{ \sigma_{\sigma, f} - \sigma_{\sigma, f+1} \}$  requires careful manipulation but no further appeal to physics. The final result can be written

$$M_f^{rL} = \max \{ 0, 1 + \langle \frac{1}{2}c \rangle \}, \quad \underline{6.16}$$

where  $\langle x \rangle$  denotes the greatest integer  $\leq x$ ,

$$c = \frac{1}{2}r - L + f + \nu - |\nu| + \beta - |\beta|,$$

$$\nu = \langle \frac{1}{2}(L - \frac{1}{2}r + f) \rangle, \text{ and } \beta = \langle \frac{1}{2}L - f \rangle.$$

Now these coefficients give the number of times a set of two-particle states to which we can attach a label  $f$ , occurs in the representation of  $SU_2$  formed by the states labelled  $r, L$ . Equivalently the numbers  $M_f^{rL}$  give the number of times the orbital angular momentum  $L$  occurs in the set of states bearing labels  $r$  and  $f$ . The latter phrasing is more relevant for what follows.

All the development sketched so far involved only the two-particle case. Now the two publications of Bargmann and Moshinsky (B60, 61) develop a classification of many-particle functions which is in fact identical to Elliott's classification. They choose to present it in terms of complicated theorems and manipulations of elementary algebra, relegating the group theory mostly to footnotes. They treat first the two-particle case obtaining a classification in terms of the two quantum numbers  $r$  and  $f$ , which we have seen label reps of  $SU_2$ . They later obtain the states of



total orbital angular momentum  $L$  for each set of values  $r, f$ , i.e. our numbers  $M_f^{rL}$ , but in a completely different way and without associating them with the reduction of Moshinsky's  $\Delta^{rL}$ . They then show that the A-particle case effectively reduces to the two-particle case, so that the A-nucleon functions are also labelled by two quantum numbers  $r, f$  ( $r$  is now the energy quantum number of the equivalent two-particle state, not that of the A-particle state, which must be remembered separately).

Their A-particle classification is in fact a classification according to reps of the group  $SU_A$ , just as their two-particle classification was an  $SU_2$  classification, because it is based on the fact that the A-particle functions are homogeneous polynomials in  $A$  variables (the  $A$  creation operators  $a_i^*$ , disregarding the  $xyz$  labels).

We now use the formalism due to Kretzschmar and reviewed at the beginning of this chapter to show why their  $SU_A$  classification must reduce in some sense to the  $SU_2$  classification, and at the same time be identical to Elliott's  $SU_3$  classification. We have seen that although the reps of  $SU_A$  are labelled by all YD's  $[\rho']$  of at most  $A$  rows, in fact only those with  $[\rho'] = [\rho]$  are acceptable, where  $[\rho]$  is a YD of at most three rows, labelling a rep of  $SU_3$ . Thus the  $SU_A$  classification is identical to the  $SU_3$  classification.

Moreover the reps  $[\rho] = [\rho_1, \rho_2, \rho_3]$  of  $SU_3$  all belong to rep classes that can be labelled by two numbers, such as Elliott's symbols  $\lambda = \rho_1 - \rho_2$   $\mu = \rho_2 - \rho_3$ . But this is equivalent to discarding all complete columns of three boxes, leaving a YD of at most two rows, i.e. the YD  $[\lambda + \mu, \mu]$  with  $\rho_3 = 0$ . In that sense the  $SU_A$  or  $SU_3$  classification resembles an  $SU_2$  classification, and we can speak of a pseudo-spin. The resemblance is superficial for although YD's of two rows can be thought of as labelling



reps of  $SU_2$ , only one parameter, say  $f = \frac{1}{2}(\rho_1 - \rho_2)$  would be significant if we were really interested in  $SU_2$ , whereas two parameters are significant for  $SU_3$ . If instead of taking the two parameters to be  $\lambda = \rho_1 - \rho_2$  and  $\mu = \rho_2$  in the YD of two rows we take them to be  $r = \rho_1 + \rho_2$  and  $f = \frac{1}{2}(\rho_1 - \rho_2)$ , we obtain the quantum numbers used by Bargmann and Moshinsky. The statement made above, that  $r$  is the energy of the corresponding two-particle state, is confirmed by the formula  $r = \rho_1 + \rho_2$ , since this gives the number of boxes of the corresponding YD of two rows. The link between the two theories is therefore summarized by the following relations:

$$\begin{aligned} r &= \lambda + 2\mu, & f &= \frac{1}{2}\lambda; \\ \text{or } \lambda &= 2f, & \mu &= \frac{1}{2}(r - 2f). \end{aligned} \tag{6.16}$$

(Our  $r$  is  $\rho$  in Moshinsky's papers.)

Bargmann and Moshinsky then introduce what we have called the  $SU_3$  model, using Moszkowski's residual interaction  $-Q^2/\mathcal{Q}$  (Section 5.2), and obtaining the eigenvalue expression

$$\frac{\hbar^2}{2\mathcal{Q}} \{ -r(r+6)/3 - 4f(f+1) + L(L+1) \} \tag{6.17}$$

From the formulas above, this is of course equal to the eigenvalue expression  $\hbar^2 B/3\mathcal{Q}$  of Section 5.2, with

$$B_0 = 2(\lambda^2 + 3\lambda + \lambda\mu + 3\mu + \mu^2) = \frac{1}{2}r(r+6) + 6f(f+1). \tag{6.18}$$

Finally the numbers  $M_I^{rL}$  give the values of  $L$  consistent with given quantum numbers  $r, f$ , for the  $A$ -particle functions as well as for the two-particle functions. But these numbers are simply the result of a subduction  $SU_3 \rightarrow SU_2$ , breaking down a rep  $r, f$  of  $SU_3$  into reps  $L$  of  $SU_2$ , and are therefore the same numbers we obtained in Section 3.3, from the out-plethysm  $[2]\overline{o}[\rho]$ . They contain the prediction of rotational bands as displayed in Table 2 of Bargmann and Moshinsky (B60). In fact these numbers have been obtained in different ways by several authors. Apart





from the formula derived in this section and the inequalities of Bargmann and Moshinsky (B61), we mention the explicit formula by Kretzschmar (K60), and that published in 1949 by Racah (Ra49), which is probably the earliest. These formulas all look different and are obtained as solutions of apparently unrelated problems.

We end with a word of caution. It should be clear from the explanations above that the fact that the pseudo-spin is mathematically analogous to a spin has nothing to do with the rotational properties of this classification, as might be thought at first sight. Indeed in 6.17, this purely mathematical resemblance is responsible for the form of the term  $f(f+1)$  in the eigenvalue expression, not at all for the term  $L(L+1)$ .

Later extensions of the formalism reviewed in this section have been given by Moshinsky (M62, 62a).



## Chapter 7      CONCLUSIONS

This final chapter attempts to summarize the main conclusions arising from various parts of the thesis, and at the same time to point out several open questions on which further work seems to be indicated.

In the hope of providing a utilitarian guide to the actual application of the representation theory of the symmetric and of the linear groups to many-particle wave functions, I have presented mathematical operations acting on the Young Diagrams themselves. The duality between symmetric and linear groups then implies that results pertaining to these different groups are easily extracted by applying different interpretations (and trivial modifications such as sifting) to the same set of results of YD operations, calculated once and for all. An interesting consequence of this is that any number of different dimensional checks can be brought to bear upon the same YD formulas. It is conjectured that these dimensional checks alone actually suffice to determine the results of the YD operations; this is subject to the verification that enough of the simultaneous equations these checks provide are linearly independent in the general case. Whether or not further work confirms this point in the general case, the computational situation is such that information so derived should be useful in combination with other techniques.

It should be instructive to reformulate this branch of mathematics in such a way as to make full use of the powerful but little-known developments recently given by Mackey (Ma51), which emphasize among other things relations between induced and subduced representations, along the lines of the theorem of Frobenius.



Considerable effort has been devoted to investigating various types of relationships between different classifications, in particular between the supermultiplet and  $SU_3$  classifications. After a review of the manner in which the supermultiplet classification can be viewed equivalently as an  $S_k$  or an  $SU_5$  classification, it is pointed out that in the  $p$  shell, not only do the supermultiplet and  $SU_3$  classifications coincide in assigning labels to states, but the supermultiplet model can be made to reproduce exactly the relative energy predictions of the  $SU_3$  model, by a suitable choice of values of the parameters  $\mathcal{L}$  and  $\mathcal{K}$ . Further, an instructive parallel is described between the homogeneous polynomials of degree  $r$  in three creation operators and the homogeneous polynomials of degree  $k$  in  $s$  single-particle functions.

Particular attention has been given to the problem of extending both the supermultiplet and the  $SU_3$  classification procedures to excited configurations, in such a way as not to disturb the physical relevance of the respective models. The techniques developed for this purpose have been applied to the spectrum of  $Ne^{20}$ , completing an application of the  $SU_3$  model to the excitation energies of all observed states in that nucleus. This application stressed those predictions of the  $SU_3$  model that do not coincide with rotational model predictions (i.e. the relative positions of the different bands). The same sort of qualitative agreement is found for these predictions not contained in the rotational model as for the predicted energy spacings within each band, using a single adjustable parameter for the entire analysis. This shows that for  $Ne^{20}$  at least the physical relevance of the  $SU_3$  model goes beyond that of the rotational model. In particular we have considered the position of the leading excited-configuration state of  $Ne^{20}$  relative to the ground state, for which the  $SU_3$  model predicts a





lowering in energy that does not seem to be accounted for by other models.

By generalizing some of the considerations of Section 6.1 which were aimed at ascertaining whether this particular  $\text{Ne}^{20}$  level is spurious, it should be possible to find simpler criteria for the elimination of spurious states in excited configurations.

In connexion with the excited-configuration techniques (including the testing of states for spuriousness), it might now be possible and interesting to see whether the conclusions of Motz and Feenberg (Mt38) on the statistical distribution of energy levels in light nuclei would be seriously modified by the removal of spurious states, and perhaps by the ordering of states according to the  $\text{SU}_3$  model. A similar investigation could also be carried out beyond the p shell.

The relation of the classification of Bargmann and Moshinsky to that of Elliott has been investigated, using developments given by Kretzschmar. The two classifications are not only related but identical. A calculation linking Moshinsky's pseudo-spin to the reduction of a representation occurring in his transformation bracket formalism was also presented.

The remarks made in Chapter 5 on the uses and limitations of  $\text{SU}_3$  theory lead to one principal conclusion. It would be most desirable to have a group-theoretical classification leading to rotational bands as Elliott's does, but in jj coupling and in the presence of spin-orbit interaction. Let us suppose this problem is approached in a way that closely parallels Elliott's handling of the LS-coupling problem. By hypothesis we want to extract typical rotational model results from the independent-particle model. This seems to force us to consider subgroups  $G$  of some group  $\text{SU}_g$ , with the rotation group  $\text{SU}_2$  contained as a subgroup in  $G$ . The group





$SU_s$  is the maximum symmetry group of the single-particle hamiltonian; as in Chapter 3,  $s$  is the degeneracy of the single-particle functions. Now in this case we want to discuss functions of space coordinates and ordinary spin together, so that for an unperturbed and spin-independent harmonic oscillator potential  $s$  would be twice the corresponding quantity in Elliott's treatment, and the rotation group  $SU_2$  mentioned should describe simultaneous rotations in space and spin, instead of space rotations only as in Elliott's work. So far there should be no difficulty. But when we introduce into the single-particle hamiltonian a term proportional to  $\underline{l} \cdot \underline{s}$ , the degeneracy  $s$  is sharply reduced to  $s = 2j+1$ . (The group  $SU_{2j+1}$  was introduced by Flowers (Fw52) as a preliminary step in his symplectic-group classification for  $jj$ -coupling states.) Now Elliott rightly considered that the key which made possible his  $SU_3$  classification was his consideration of 'mixed configurations', i.e. mixing the several values of  $\ell$  that are degenerate for a pure harmonic oscillator. Here the trouble is that the spin-orbit interaction apparently prevents us from mixing several values of  $j$ . Some of these statements however may be too restrictive and the subject requires further consideration.

It may also be possible to tackle the problem from the other end: having understood (by plethysm techniques or otherwise) how the subduction  $SU_3 \rightarrow SU_2$  introduces rotational bands in LS coupling, it may be possible to formulate restrictions that must be obeyed by the group  $G$  if the subduction  $G \rightarrow SU_2$  is to yield rotational bands in  $jj$  coupling.

Further work also remains to be done if a satisfactory understanding is to be obtained of the relations between  $SU_3$  theory and other recent developments such as those given by Peierls and Thouless (P62 and references therein).



To summarize, I believe this thesis illustrates in various ways that  $SU_3$  theory, despite the limitations pointed out, has provided and will continue to provide both a useful tool in the detailed investigation of nuclear spectra, and a useful point of view for the study of the relations between more elaborate models.

I would like to end this long and rather technical thesis by a non-technical concluding impression. This type of investigation is motivated partly by the usual effort to explain experimental results and confirm the explanation by improving agreement with other experimental results, but motivated even more by an effort to discover relationships between existing independent explanations, so as to achieve a deeper understanding of the existing explanations. As such it forces one to realize to what extent physics depends on our arbitrarily chosen methods of description as well as on the properties of the external world it describes. This realization is certainly one of the recurring themes of current scientific thought.



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